

Cox rings of some symplectic resolutions of quotient singularities

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ABSTRACT. We investigate Cox rings of symplectic resolutions of quotients of \mathbb{C}^n by finite symplectic group actions. We generalize the main result of [DBW14]. We propose a finite generating set of the Cox ring of a symplectic resolution and prove that under a condition concerning monomial valuations it is sufficient. Also, three 4-dimensional examples are described in detail. Generators of the (expected) Cox rings of symplectic resolutions are computed and in one case a resolution is constructed as a GIT quotient of the spectrum of the Cox ring.

1. Introduction

This article is the second step towards understanding the total coordinate rings of resolutions of higher dimensional quotient singularities. It follows [DBW14], where the case of a certain 32-element symplectic group G is considered: first a generating set of a ring expected to be the Cox ring of a symplectic resolution of \mathbb{C}^4/G is given, and then it is used to construct all 81 symplectic resolutions of this singularity. The existence of symplectic resolutions of \mathbb{C}^4/G was proven before by Bellamy and Schedler, see [BS13b], but no construction of such a resolution was known. Earlier, surface quotient singularities were used as a test case for developing ideas related to describing the Cox rings of resolutions and reconstructing these resolutions via Cox rings, see [DB13] and also [FGAL11]. Similar concepts, now classical for toric varieties, were also studied for varieties with complexity one torus action, see [ADHL14].

The aim of the present paper is further development of ideas presented in [DBW14]. First, we study general properties of the Cox ring $\mathcal{R}(X)$ of a resolution X of a quotient of $V \simeq \mathbb{C}^n$ by an action of a finite matrix group G . We attempt to describe them by giving a (finite) generating set in a simpler ring: the tensor product of the Cox ring of the singularity V/G , which by [AG10] is isomorphic to the ring of invariants $\mathbb{C}[V]^{[G,G]}$, and the coordinate ring of the Picard torus of the resolution.

We restrict ourselves to symplectic singularities and their symplectic, i.e. crepant by [Ver00, Thm. 2.5], resolutions. This allows us to benefit from additional information on the structure of resolutions, in particular the McKay correspondence, proven in the symplectic case by Kaledin in [Kal02]. The language of monomial valuations, used therein to relate divisors in the resolution to conjugacy classes in

the group as suggested by Ito and Reid in [IR96], is also present in our description of the generating set of the Cox ring. Our main theoretical result is (cf. [DBW14, Thm. 1.1])

THEOREM (3.8). *Let G be a finite symplectic group generated by symplectic reflections, such that the commutator subgroup $[G, G]$ does not contain any symplectic reflections. In addition, assume that condition (3.6), concerning monomial valuations on the rational function field of $\mathbb{C}[V]^{[G, G]}$, is satisfied. Let t_1, \dots, t_m denote the coordinates on the Picard torus \mathbb{T} of X . Then the Cox ring $\mathcal{R}(X)$ of any symplectic resolution X of V/G is generated by*

- $t_1^{E_i \cdot C_1} \dots t_m^{E_i \cdot C_m}$ for $i = 1, \dots, m$, and
- $\bar{\phi}_j = \phi_j \cdot \prod_{i=1}^m t_i^{\bar{D}_j \cdot C_i}$ for $j = 1, \dots, n$,

where ϕ_j are eigenvectors of the action of $\text{Ab}(G) = G/[G, G]$ on $\mathbb{C}[V]^{[G, G]}$, \bar{D}_j are strict transforms of corresponding divisors on V/G , E_i are exceptional divisors of the resolution and C_i are curves which are generic fibers of the resolution restricted to E_i .

The general part is followed by a detailed study of three examples in dimension 4. In all of them we compute the generating set of the Cox ring proposed in Theorem 3.8. The simplest one is the case of the symmetric group S_3 . Considered representation contains only one conjugacy class of symplectic reflections, i.e. just one monomial valuation is involved, hence condition (3.6) is relatively easy to check. In this case there is only one symplectic resolution.

For the wreath product $\mathbb{Z}_2 \wr S_2$, isomorphic to the dihedral group D_8 , the proof of condition (3.6) requires an additional element. Since there are two conjugacy classes of symplectic reflections, one has to study two monomial valuations and the relation between them. In this case there are two symplectic resolution of \mathbb{C}^4/G , which differ by a flop. We describe their geometry, in particular the structure of the central fiber, by analyzing directly the Picard torus action on the spectrum of their Cox ring and its GIT quotients.

In the last example we tackle the case of the reducible representation of the binary tetrahedral group, which will be denoted here by G_4 . However, the proposed set of the Cox rings generators turns out to be too complex to prove condition (3.6) for this group.

In section 1.2 we explain how these examples fit in the general scheme of knowledge about the existence of symplectic resolution of symplectic quotient singularities.

1.1. Outline of the paper. In section 2 we recall basic information about the Cox rings after [ADHL14] and summarize the results of section 3 of [DBW14], which gives the setting for describing the generators of Cox rings of resolutions of quotient singularities. Then, sections 3.1 and 3.2 are devoted to the precise formulation of Theorem 3.8 together with additional assumption (3.6), and section 3.3 contains its proof.

The examples part is started by the case of S_3 : the generators from Theorem 3.8 are computed in section 4.1 and in section 4.2 we prove that they really generate the Cox ring of the resolution. The generators for $\mathbb{Z}_2 \wr S_2$ are described in section 5.1

and we sketch parts of the proof that we indeed obtain the Cox ring of a symplectic resolution (a complete argument will be included in [DB15]). Then we investigate the geometry of the quotients of the spectrum of this ring: we embed it in an affine space in section 5.3, analyze stability and prove the smoothness of certain GIT quotients in sections 5.4, 5.5 and in section 5.6 the structure of the central fiber is described. Finally, in section 6 the proposed generators of the Cox ring for G_4 are given.

1.2. Existence of symplectic resolutions of quotient singularities. As proved by Verbitsky in [Ver00], if a symplectic resolution of \mathbb{C}^n/G exists then G is generated by symplectic reflections, i.e. elements which fix a subspace of codimension 2. Note that this is a necessary but insufficient criterion.

Finite symplectically irreducible groups generated by symplectic reflections are classified by Cohen in [Coh80]. The problem of the existence of symplectic resolutions was investigated in [GK04, Bel09, BS13a, BS13b]. According to [Bel09] and [BS13b, 4.1], for the following (symplectically irreducible) $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$ it is known that the symplectic resolution exists.

- (1) S_{n+1} acting on \mathbb{C}^{2n} via direct sum of two copies of standard n -dimensional representation of S_{n+1} . Here the resolution can be constructed via the Hilbert scheme $\mathrm{Hilb}^{n+1}(\mathbb{C}^2)$. The example of S_3 from section 4 is the (only) 4-dimensional representative of his family.
- (2) $H \wr S_n = H^n \rtimes S_n$, where H is a finite irreducible subgroup of $\mathrm{SL}_2(\mathbb{C})$, so that \mathbb{C}^2/H is one of the du Val singularities, and S_n acts on H^n by permutations of coordinates. The natural product representation of H^n on \mathbb{C}^{2n} extend naturally to the action of $H \wr S_n$. The resolution is $\mathrm{Hilb}^n(S)$ where S is a minimal resolution of the du Val singularity \mathbb{C}^2/H . The example of $\mathbb{Z}_2 \wr S_2$ analyzed in section 5 belongs to this family, in particular we show how to describe the resolution as a GIT quotient of the spectrum of its Cox ring.
- (3) A representation $G_4 \subset \mathrm{Sp}_4(\mathbb{C})$ of the binary tetrahedral group. The existence of a symplectic resolution was first proved by Bellamy in [Bel09] and constructed later by Lehn and Sorger, see [LS12]. We consider this case in section 6.
- (4) A group of order 32 acting on \mathbb{C}^4 , for which the existence of symplectic resolutions was proved in [BS13a], and their construction and the Cox ring were investigated in [DBW14].

Apart from this list, [Bel09] and [BS13b] give also negative results. The non-existence of symplectic resolutions is shown for certain infinite families of representations from the Cohen's classification [Coh80]. On the other hand, even in dimension 4 there still are infinitely many cases for which the question of existence of the symplectic resolution remains unanswered.

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2. Setting

The aim of this section is to recall the general setting for the construction of Cox rings of symplectic resolutions of quotient singularities – we summarize the results of [DBW14, Sect. 2.D, 3.A, 3.B].

Let $V = \mathbb{C}^{2n}$. We consider quotients $Y = V/G$, where $G \subset \mathrm{Sp}(V)$ is a finite subgroup of linear transformations of V preserving a symplectic form on V . Assume that a symplectic resolution of singularities

$$\varphi: X \rightarrow Y$$

exists. Our goal is to determine the generators of the Cox ring of X , and then to use this ring to recover the construction of the resolution.

Recall that the Cox ring, or the total coordinate ring, of a normal, irreducible complex algebraic variety Z is the module

$$\mathcal{R}(Z) = \bigoplus_{[D] \in \mathrm{Cl}(Z)} \Gamma(Z, \mathcal{O}_Z(D))$$

with multiplication coming from the identification

$$\Gamma(Z, \mathcal{O}_Z(D)) = \{f \in \mathbb{C}(Z)^* : \mathrm{div}(f) + D \geq 0\} \cup \{0\}$$

of global sections with rational functions on Z . If the class group of Weil divisors $\mathrm{Cl}(Z)$ is finitely generated and free, as by [DBW14, Lem. 2.13] it happens in the case of resolutions of quotient singularities, it is enough to fix inclusions $\Gamma(Z, \mathcal{O}_Z(D_i)) \subset \mathbb{C}(Z)$ for chosen representatives D_i of a basis $\{[D_i] : i = 1, \dots, n\}$ of $\mathrm{Cl}(Z)$, and multiply sections as corresponding rational functions. Otherwise, relations in $\mathrm{Cl}(Z)$ have to be taken into account, for the details see [ADHL14, Sect. 1.4].

Assume that $\mathrm{Cl}(Z)$ is finitely generated and free. If $\mathcal{R}(Z)$ is finitely generated \mathbb{C} -algebra (thus Z is a Mori Dream Space), there is an action of the torus $\mathbb{T}_{\mathrm{Cl}(Z)} = \mathrm{Hom}(\mathrm{Cl}(Z), \mathbb{C}^*)$ on its spectrum, coming from the $\mathrm{Cl}(Z)$ grading. If in addition Z is projective over $\mathbb{C}[Z]$, we may recover Z , and its other birational models, as GIT quotients of $\mathrm{Spec} \mathcal{R}(Z)$ by $\mathbb{T}_{\mathrm{Cl}(Z)}$. This is the main idea behind resolving quotient singularities via Cox rings: if $\mathcal{R}(Z)$ can be found just based on the group structure data and general information on the resolution, one may try to construct new resolutions as GIT quotients of $\mathrm{Spec} \mathcal{R}(Z)$.

2.1. Embedding of the Cox ring of a resolution. By $\mathcal{R}(X)$ and $\mathcal{R}(Y)$ we denote the Cox rings of the resolution X and the singular quotient Y respectively. One can describe the class groups of X and Y , by which these rings are graded.

PROPOSITION 2.1. *By [Ben93, Prop. 3.9.3] and [DBW14, Lem. 2.13, 2.14] we have:*

- (a) $\mathrm{Cl}(Y)$ is naturally isomorphic to the group $G^\vee = \mathrm{Hom}(G, \mathbb{C}^*) = \mathrm{Ab}(G)^\vee$,
- (b) $\mathrm{Cl}(X) = \mathrm{Pic}(X) \simeq \mathbb{Z}^m$, where m is the number of irreducible components of the exceptional divisor, which by the McKay correspondence (see [Kal02]) is equal to the number of conjugacy classes of symplectic reflections in G .

We have a natural surjective map $\varphi_*: \text{Cl}(X) \rightarrow \text{Cl}(Y)$ induced by the push-forward of Weil divisors via φ . The resolution φ induces also a natural injective homomorphism $\Gamma(X, \mathcal{O}_X(D)) \rightarrow \Gamma(Y, \mathcal{O}_Y(\varphi_*D))$ of modules of global sections of rank 1 reflexive sheaves associated with a Weil divisor D on X and its push-forward $\varphi_*(D)$, see [DBW14, 3.A]. Altogether we obtain a homomorphism of graded rings

$$\varphi_*: \mathcal{R}(X) \rightarrow \mathcal{R}(Y).$$

There is also a comultiplication homomorphism

$$\mathcal{R}(X) \rightarrow \mathcal{R}(X) \otimes \mathbb{C}[\text{Cl}(X)],$$

given by the action of the Picard torus $\mathbb{T}_{\text{Cl}(X)}$ on $\text{Spec } \mathcal{R}(X)$, compatible with the $\text{Cl}(X)$ -grading. On the level of graded pieces of $\mathcal{R}(X)$ it is given by

$$\Gamma(X, \mathcal{O}_X(D)) \ni f \mapsto f \otimes \chi^D \in \Gamma(X, \mathcal{O}_X(D)) \otimes \mathbb{C}[\text{Cl}(X)],$$

where χ^D is a character of torus $\mathbb{T}_{\text{Cl}(X)}$ corresponding to the class of D in $\text{Cl}(X)$.

PROPOSITION 2.2 ([DBW14, Prop. 3.8]). *The composition of the comultiplication homomorphism and $\varphi_* \otimes \text{id}$*

$$\Theta: \mathcal{R}(X) \rightarrow \mathcal{R}(X) \otimes \mathbb{C}[\text{Cl}(X)] \rightarrow \mathcal{R}(Y) \otimes \mathbb{C}[\text{Cl}(X)]$$

is an injective graded ring homomorphism.

By [AG10, Thm. 3.1] (see also [DB13, Lem. 6.2]) the Cox ring $\mathcal{R}(Y)$ can be viewed as the ring of invariants $\mathbb{C}[V]^{[G,G]} \subset \mathbb{C}[V]$ of commutator subgroup $[G, G]$, with grading given by the induced action of $\text{Ab}(G)$ on $V/[G, G]$. Hence by Prop. 2.2 we may identify $\mathcal{R}(X)$ with a subring of a better understood ring:

$$(2.3) \quad \mathcal{R}(X) \subseteq \mathbb{C}[V]^{[G,G]} \otimes \mathbb{C}[\text{Cl}(X)].$$

Thus, the task of determining $\mathcal{R}(X)$ is reduced to pointing out a set of elements of $\mathbb{C}[V]^{[G,G]} \otimes \mathbb{C}[\text{Cl}(X)]$ which generate $\mathcal{R}(X)$ as a \mathbb{C} -algebra.

2.2. Describing the image of Θ . Let E_1, \dots, E_m be the components of exceptional divisor of the resolution $\varphi: X \rightarrow Y$. Let C_i denote the generic fiber of $\varphi|_{E_i}: E_i \rightarrow \varphi(E_i)$. As it was noted in [DBW14, Sect. 2.D], classes of C_1, \dots, C_m form a basis of the vector space $N_1(X/Y)$. The dual basis of $N^1(X/Y)$ (via the intersection pairing) will be denoted by L_1, \dots, L_m . Then the coefficients of a divisor D on X in this basis are the intersection numbers (C_i, D) .

PROPOSITION 2.4 ([DBW14, 2.16]). *We have a commutative diagram, whose rows are exact sequences:*

$$(2.5) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \bigoplus_{i=1}^m \mathbb{Z}E_i & \longrightarrow & \text{Cl}(X) & \xrightarrow{\varphi_*} & \text{Cl}(Y) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \bigoplus_{i=1}^m \mathbb{Z}E_i & \longrightarrow & \bigoplus_{i=1}^m \mathbb{Z}L_i & \longrightarrow & Q \longrightarrow 0, \end{array}$$

Here the homomorphism $\bigoplus_i \mathbb{Z}E_i \rightarrow \bigoplus_i \mathbb{Z}L_i$ takes E_i to $\sum_j (E_i, C_j) L_j$, and the group Q is defined as its cokernel. The image of $D \in \text{Cl}(Y)$ in Q is given by $D \mapsto \sum_i (\varphi_*^{-1}(D), C_i) [L_i]$.

It is known that variety V/G has du Val singularities in codimension 2 and the intersection matrix $(E_i.C_j)$ is a direct sum of corresponding Cartan matrices, see [WW03, Thm. 1.3] and [AW14, Thm. 1.4]. In particular, $(E_i.C_j)$ is invertible and so the lattice $\text{Cl}(X) = \text{Pic}(X) \subset N^1(X/Y)$ is a finite index sublattice of the lattice $\bigoplus_{i=1}^m \mathbb{Z}L_i \subset N^1(X/Y)$.

REMARK 2.6. Note that by Proposition 2.4 we have $\text{Cl}(X) = \bigoplus_i \mathbb{Z}L_i$ if and only if the index of subgroup $\bigoplus_i \mathbb{Z}E_i \subseteq \bigoplus_i \mathbb{Z}L_i$ is equal to the order of $\text{Cl}(Y) \cong \text{Ab}(G) = G/[G, G]$. The index of $\bigoplus_i \mathbb{Z}E_i \subseteq \bigoplus_i \mathbb{Z}L_i$ is the absolute value of the determinant of the intersection matrix $(E_i.C_j)$. In the case of the 32-element group considered in [DBW14] the order of $\text{Ab}(G)$ is smaller than $|\det(E_i.C_j)_{i,j}|$: there $\text{Ab}(G) \cong \mathbb{Z}^4$ but $(E_i.C_j)_{i,j}$ is the direct sum of 5 copies of (-2) . However, it is the only case in dimension 4 where it is known that a symplectic resolution exists and $|\text{Ab}(G)| < |\det(E_i.C_j)|$. This can be verified via direct inspection of 4-dimensional cases from the list in section 1.2.

- For groups of the form $H \wr S_2$, where $H \subset \text{SL}_2(\mathbb{C})$, the intersection matrix is a direct product of two Cartan matrices: one of A_1 type, i.e. (-2) , and C_H corresponding to the singularity \mathbb{C}^2/H . We have $\text{Ab}(H \wr S_2) = \text{Ab}(H) \times S_2$. The claim follows from the observation that $|\text{Ab}(H)| = |\det C_H|$.
- For the group S_3 we have $|\text{Ab}(G)| = 2$ and the corresponding Cartan matrix is (-2) .
- For the group G_4 we have $|\text{Ab}(G)| = 3$ and $(E_i.C_j)$ is A_2 -type Cartan matrix, so $\det(E_i.C_j) = 3$.

In particular, in examples given in sections 4-6 the diagram (2.5) reduces to one row.

If we embed $\mathbb{C}[\text{Cl}(X)]$ into $\mathbb{C}[\bigoplus_i \mathbb{Z}L_i]$ and denote by t_i the variable corresponding to $[L_i]$, the homomorphism Θ introduced in Proposition 2.2 becomes the injective graded ring homomorphism

$$(2.7) \quad \overline{\Theta} : \mathcal{R}(X) \rightarrow \mathcal{R}(Y)[t_1^{\pm 1}, \dots, t_m^{\pm 1}].$$

In particular, we may identify $\mathcal{R}(X)$ with its image $\overline{\Theta}(\mathcal{R}(X))$.

Note that by Remark 2.6 in cases S_3 , $\mathbb{Z}_2 \wr S_2$, G_4 (see sections 4-6) we have $\text{Cl}(X) = \bigoplus_i \mathbb{Z}L_i$, that is $\overline{\Theta} = \Theta$.

To describe the elements of the image of $\overline{\Theta}$ we will use the following convention (see [DBW14, Sect. 2.A]).

NOTATION 2.8. Let D be a Weil divisor on a variety Z and let $\mathcal{O}(D)$ be the associated reflexive sheaf. For each section in $\Gamma(V, \mathcal{O}(D)) = \{f \in K(Z)^* : \text{div } f + D \geq 0\} \cup \{0\}$ one may take the corresponding effective divisor $D' = \text{div } f + D$. We denote the section corresponding to the divisor D' by $f_{D'}$, it is determined up to the multiplication by a constant. If $Z' \rightarrow Z$ is a birational map then we will denote the strict transform of D by \overline{D} .

The next result, cf. [DBW14, Cor. 3.11], follows directly from the definition of $\overline{\Theta}$.

PROPOSITION 2.9. *If $D \in \text{Div}(X)$ and $f_D \in \mathcal{R}(X)$, then*

$$(2.10) \quad \overline{\Theta}(f_D) = f_{\varphi_* D} \cdot t_1^{D.C_1} \dots t_m^{D.C_m}.$$

2.3. Monomial valuations. We will need the concept of monomial valuations, which are used e.g. in [Kal02] in the context of the McKay correspondence for resolutions of quotient symplectic singularities. For each positive integer r fix a root of unity $\epsilon_r \in \mathbb{C}^*$ of order r . Take a group monomorphism $\mu_r = \langle \epsilon_r \rangle \rightarrow \mathrm{GL}_n(\mathbb{C})$. This is equivalent to fixing a linear operator $T \in \mathrm{GL}_n(\mathbb{C})$ of order r , the image of the chosen generator ϵ_r . Assume that T acts on \mathbb{C}^n with coordinates x_1, \dots, x_n by a diagonal matrix $\mathrm{diag}(\epsilon_r^{a_1}, \dots, \epsilon_r^{a_n})$ where $0 \leq a_i < r$.

DEFINITION 2.11. The monomial valuation $\nu_T : \mathbb{C}(x_1, \dots, x_n) \rightarrow \mathbb{Z}$ is defined by

$$\nu_T \left(\sum_{\alpha} c_{\alpha} x^{\alpha} \right) = \min \{ \langle (a_1, \dots, a_n), \alpha \rangle : c_{\alpha} \neq 0 \},$$

where $\langle \alpha, \beta \rangle = \sum_i \alpha_i \beta_i$.

By the McKay correspondence each component E_i of the exceptional divisor of the resolution φ corresponds to a conjugacy class of symplectic reflections. We fix representatives of conjugacy classes: T_i represents the i -th class. Let r_i be the order of T_i . We will write ν_i for ν_{T_i} . The next lemma will be useful in relating values of ν_i to the intersection numbers of divisors with curves C_i .

LEMMA 2.12 ([Kal02, Lem. 2.5]). $\nu_i|_{\mathbb{C}(V)^G} = r_i \nu_{E_i}$, where ν_{E_i} is the divisorial valuation centered at E_i .

3. General results

Here we explain our strategy for finding generators of the Cox ring $\mathcal{R}(X)$. First, in 3.1 we propose the generating set and introduce the technical assumption on monomial valuations under which Theorem 3.8 is then proved in 3.3. The proof relies on the relation between monomial valuations and intersections with curves on symplectic resolution, described in 3.2.

REMARK 3.1. Throughout this section we will always assume that the commutator subgroup $[G, G]$ does not contain symplectic reflections. This guarantees that orders of symplectic reflections in G are the same as orders of their images in $\mathrm{Ab}(G)$.

REMARK 3.2. As we noted in 2.2, the symplectic quotient singularity V/G have codimension 2 du Val singularities. However, under the assumption that $[G, G]$ does not contain symplectic reflections V/G can have only codimension 2 singularities of A_n -type. Indeed, each of those singularities will occur along the image of subspace fixed by some symplectic reflection. Then the symplectic reflections preserving this subspace form naturally a subgroup of $\mathrm{SL}_2(\mathbb{C})$. The commutator of this subgroup does not contain symplectic reflections if and only if it is trivial, i.e. this subgroup is cyclic.

3.1. A candidate for a generating set. To simplify the notation for the Cox ring of $Y = V/G$ we set $\mathcal{P} = \mathcal{R}(Y) \simeq \mathbb{C}[V]^{[G, G]}$. We look at the $\mathrm{Cl}(Y)$ -grading on \mathcal{P} , i.e. the decomposition into eigenspaces of the $\mathrm{Ab}(G)$ action, and fix a finite generating set of \mathcal{P} .

NOTATION 3.3. By $\{\phi_1, \dots, \phi_n\}$ we denote a finite set of generators of \mathcal{P} consisting of chosen eigenvectors of the $\mathrm{Ab}(G)$ action. We think of each ϕ_j as of a global

section of a reflexive sheaf on Y and denote its divisor of zeroes by D_j and their strict transforms by \overline{D}_j .

We lift elements $\phi_j \in \mathcal{P}$ to $\mathcal{P}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ as follows:

$$\overline{\phi}_j = \phi_j \cdot \prod_{i=1}^m t_i^{\overline{D}_j \cdot C_i}.$$

The aim of this section is to prove that in some situations elements $\overline{\phi}_j$ together with certain characters of the Picard torus form a (finite) generating set of $\mathcal{R}(X)$, see Theorem 3.8. However, we need first to describe an additional assumption under which the result holds.

Let $\kappa: \mathbb{C}[w_1, \dots, w_n] \rightarrow \mathcal{P}$ be the surjective ring homomorphism sending $w_i \mapsto \phi_i$. We will need to compare monomial valuations ν_i on \mathcal{P} corresponding to symplectic reflections T_i with certain monomial valuations on $\mathbb{C}[w_1, \dots, w_n]$. To define them we lift each T_i to a linear map $\tilde{T}_i: \mathbb{C}^n \rightarrow \mathbb{C}^n$

$$\tilde{T}_i(w_1, \dots, w_n) = \left(\frac{T_i(\phi_1)}{\phi_1} w_1, \dots, \frac{T_i(\phi_n)}{\phi_n} w_n \right),$$

which is well-defined since ϕ_j are eigenvectors of $\text{Ab}(G)$.

DEFINITION 3.4. By $\tilde{\nu}_i$ we understand the monomial valuation corresponding to \tilde{T}_i .

Let r_i be the order of the image of symplectic reflection T_i , corresponding to exceptional divisor E_i , in G . Since $[G, G]$ does not contain symplectic reflections, r_i is also the order of the class of T_i in $\text{Ab}(G)$. Note that \tilde{T}_i also has order r_i .

REMARK 3.5. Let ϵ_{r_i} be the r_i -th root of unity, fixed in the definition of monomial valuations in section 2.3. Then $\frac{T_i(\phi_j)}{\phi_j} = \epsilon_r^{a_{ij}}$ where $0 \leq a_{ij} < r_i$, that is $\tilde{\nu}_i(w_j) = a_{ij}$. By the definition of a monomial valuation we see that $\nu_i(\phi_j) \geq 0$, $\nu_i(\phi_j) \equiv a_{ij} \pmod{r_i}$. Hence, in particular, $\nu_i(\phi_j) \geq \tilde{\nu}_i(w_j)$.

In the proof of Theorem 3.8 we assume that every homogeneous $f \in \mathcal{P}$ can be lifted through κ compatibly with valuations ν_i and $\tilde{\nu}_i$. More precisely,

$$(3.6) \quad \text{for any homogeneous } f \in \mathcal{P} \text{ there is } \tilde{f} \in \mathbb{C}[w_1, \dots, w_n] \text{ such that } \kappa(\tilde{f}) = f \text{ and } \nu_i(f) \leq \tilde{\nu}_i(\tilde{f}) \text{ for all } i.$$

Note that the other inequality is automatic, it follows directly from the construction of $\tilde{\nu}_i$ and the definition of monomial valuations.

REMARK 3.7. The motivation for introducing this condition comes from the embedding of the singularity and its resolution in toric varieties, see [DB13, Sect. 4.2] for the surface case. The ring $\mathbb{C}[w_1, \dots, w_n]$ is naturally identified with the Cox ring of the toric quotient $\mathbb{C}^n / \text{Ab}(G)$. This quotient is singular along fixed subspaces of \tilde{T}_i and one may construct a partial toric resolution, smooth over generic point of these subspaces. Its Cox ring can be described based on the toric data, and it is expected to have surjective homomorphism to \mathcal{R} . Thus the idea of proving the inclusion $\mathcal{R}(X) \subseteq \mathcal{R}$ is, roughly speaking, as follows: take an element of $\mathcal{R}(X)$, map it to \mathcal{P} , lift to $\mathcal{R}(\mathbb{C}^n / \text{Ab}(G)) \simeq \mathbb{C}[w_1, \dots, w_n]$, and use the description of the Cox ring of the partial toric resolution of $\mathbb{C}^n / \text{Ab}(G)$ to show a presentation in terms of

generators of \mathcal{R} . Here condition (3.6) is necessary to ensure that in this process we obtain elements of suitable graded pieces of \mathcal{R} . However, it turns out that a direct reference to toric arguments can be avoided, see the proof of Proposition 3.13.

Now we can formulate the main result of this section.

THEOREM 3.8. *Suppose that G is such that $[G, G]$ does not contain symplectic reflections and assume that condition (3.6) holds. Then $\overline{\Theta}(\mathcal{R}(X))$ is generated by*

- $t_1^{E_i \cdot C_1} \cdots t_m^{E_i \cdot C_m}$ for $i = 1, \dots, m$, and
- $\overline{\phi}_j = \phi_j \cdot \prod_{i=1}^m t_i^{\overline{D}_j \cdot C_i}$ for $j = 1, \dots, n$.

The proof requires some preparation and will be given in section 3.3.

REMARK 3.9. Every effective divisor on X can be written as $\overline{D} + \sum_i a_i E_i$ with a_i non-negative integers and \overline{D} being a strict transform of an effective Weil divisor D on V/G . In particular the class group $\text{Cl}(X) = \text{Pic}(X)$ is generated by classes of effective divisors \overline{D} together with components E_i of the exceptional divisor.

By proposition 2.9 elements of the form $t_1^{E_i \cdot C_1} \cdots t_m^{E_i \cdot C_m}$ are images of sections of line bundles $\mathcal{O}_X(E_i)$, and the elements $\phi_j \cdot t_1^{\overline{D}_j \cdot C_1} \cdots t_m^{\overline{D}_j \cdot C_m}$ are images of sections of line bundles $\mathcal{O}_X(\overline{D}_j)$. In consequence, the generating set of $\mathcal{R}(X)$ proposed in Theorem 3.8 allows us to obtain elements of all graded pieces.

Condition (3.6) is in general not easy to show. In section 4.2 it is verified in the case $G = S_3$. In section 5.2 some elements of the proof for $G = \mathbb{Z}_2 \wr S_2$ are explained, and a complete argument will be given in [DB15].

3.2. Describing intersections in terms of monomial valuations. In the proof of Theorem 3.8 we will use the language of monomial valuations, see 2.3. Here we establish the relation between monomial valuations corresponding to symplectic reflections in G and intersection numbers of divisors on X with curves C_1, \dots, C_m .

Let D be a Weil divisor on Y and $f_D \in \mathcal{P}$ be the corresponding section, see Notation 2.8. We express the intersection numbers $\overline{D} \cdot C_i$ by monomial valuations $\nu_i(f_D)$. This is used later to give a different description of proposed generators of $\overline{\Theta}(\mathcal{R}(X))$.

Set $r = \text{lcm}\{r_1, \dots, r_m\}$. Then $f_D^r \in \mathbb{C}[V]^G \subset \mathbb{C}(V)^G = \mathbb{C}(X)$ and we can take the associated principal divisor on X :

$$0 \sim \text{div}_X f_D^r = r\overline{D} + \sum_i r\nu_{E_i}(f_D)E_i.$$

Using Lemma 2.12 and intersecting with C_j we get

$$(3.10) \quad r\overline{D} \cdot C_j = - \sum_i \frac{r}{r_i} \nu_i(f_D) \cdot (E_i \cdot C_j).$$

REMARK 3.11. The intersection matrix $(E_i \cdot C_j)_{i,j}$ is invertible as a direct sum of Cartan matrices. If U is its inverse then we may rewrite (3.10) as

$$(3.12) \quad \left(\frac{1}{r_1} \nu_1(f_D), \dots, \frac{1}{r_m} \nu_m(f_D) \right) = -(\overline{D} \cdot C_1, \dots, \overline{D} \cdot C_m) \cdot U.$$

Hence for the strict transforms of Weil divisors on V/G the information encoded by monomial valuations ν_i and by the intersections with curves C_i is the same.

3.3. The image of the Cox ring is generated by indicated elements.

We denote the set consisting of elements listed in Theorem 3.8 by \mathcal{S} . Let \mathcal{R} be the subring of $\mathcal{P}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ generated by \mathcal{S} . The nontrivial part of Theorem 3.8 is to show that elements of \mathcal{S} are sufficient to generate $\overline{\Theta}(\mathcal{R}(X))$.

The next proposition is the major step toward proving Theorem 3.8. In particular, it implies that \mathcal{S} is sufficient to generate sections of all line bundles $\mathcal{O}_X(\overline{D})$ for Weil divisors D on V/G , see also Remark 3.9.

PROPOSITION 3.13. *Assume that condition (3.6) holds. Let $f = f_D \in \mathcal{P}$ be a homogeneous element with zero-divisor D . Then $f_D \cdot \prod_i t_i^{\overline{D} \cdot C_i} \in \mathcal{R}$.*

PROOF. Let $M = (\overline{D}_i \cdot C_j)_{i,j}$ for $i = 1, \dots, n$, $j = 1, \dots, m$ be the intersection matrix of strict transforms of divisors corresponding to ϕ_i with curves C_j . By Proposition 2.9 this is the (transposed) matrix of weights of the Picard torus action on ϕ_i . Also, $I = (i_1, \dots, i_m) \in \mathbb{Z}^m$ and $K = (k_1, \dots, k_n) \in \mathbb{Z}^n$ will be multiindices. Take a lift $\tilde{f} = \tilde{f}_D \in \mathbb{C}[w_1, \dots, w_n]$ of f through κ compatible with valuations, which exists by condition (3.6). Set $F = \tilde{f}(\overline{\phi}_1, \dots, \overline{\phi}_n)$.

We divide \tilde{f} into parts which in F go to elements homogeneous in variables t_1, \dots, t_m . That is, $\tilde{f} = \sum_I \overline{\beta}_I$, where

$$\overline{\beta}_I = \sum_{\{K: KM=I\}} \alpha_K \cdot w_1^{k_1} \cdots w_n^{k_n}.$$

Let $\beta_I = \kappa(\overline{\beta}_I) = \overline{\beta}_I(\phi_1, \dots, \phi_n)$. Note that $\beta_I \prod_j t_j^{i_j} = \overline{\beta}_I(\overline{\phi}_1, \dots, \overline{\phi}_n)$ is an element of \mathcal{R} . Now we may write

$$f \cdot \prod_j t_j^{\overline{D} \cdot C_j} = \kappa(\tilde{f}) \cdot \prod_j t_j^{\overline{D} \cdot C_j} = \left(\sum_I \kappa(\overline{\beta}_I) \right) \cdot \prod_j t_j^{\overline{D} \cdot C_j} = \sum_I \left(\beta_I \prod_j t_j^{i_j} \right) \cdot \prod_j t_j^{\overline{D} \cdot C_j - i_j}.$$

To show that the right hand side is indeed in \mathcal{R} it suffices to prove that for any multiindex I such that $\beta_I \neq 0$ we have $\prod_j t_j^{\overline{D} \cdot C_j - i_j} \in \mathcal{R}$.

We will show how to get non-negative integers a_1, \dots, a_m such that

$$\prod_{j=1}^m t_j^{\overline{D} \cdot C_j - i_j} = \left(\prod_{i=1}^n t_i^{E_1 \cdot C_i} \right)^{a_1} \cdots \left(\prod_{i=1}^n t_i^{E_m \cdot C_i} \right)^{a_m}.$$

This will end the proof since factors of the right hand side are in the chosen generating set \mathcal{S} of \mathcal{R} .

This means that $(a_1, \dots, a_m) = (\overline{D} \cdot C_1 - i_1, \dots, \overline{D} \cdot C_m - i_m) \cdot U$, where U is the inverse of the Cartan matrix for considered singularity. By (3.12) applied to \overline{D} we get $(\overline{D} \cdot C_1, \dots, \overline{D} \cdot C_m) \cdot U = -\left(\frac{1}{r_1} \nu_1(f), \dots, \frac{1}{r_m} \nu_m(f)\right)$.

Take any $K = (k_1, \dots, k_m)$ such that $KM = I$ and $\alpha_K \neq 0$. Now (3.12) applied to $\sum k_i \overline{D}_i$ gives $IU = KMU = -\left(\frac{1}{r_i} \nu_1(\phi_1^{k_1} \cdots \phi_n^{k_n}), \dots, \frac{1}{r_m} \nu_m(\phi_1^{k_1} \cdots \phi_n^{k_n})\right)$.

That is, we may take $a_j = \frac{1}{r_j}(\nu_j(\phi_1^{k_1} \cdots \phi_n^{k_n}) - \nu_j(f))$, as soon as we show that they are non-negative integers. For the non-negativity observe that

$$\nu_j(f) \leq \tilde{\nu}_j(\tilde{f}) \leq \tilde{\nu}_j(w_1^{k_1} \cdots w_n^{k_n}) \leq \nu_j(\phi_1^{k_1} \cdots \phi_n^{k_n}),$$

where the first inequality is a consequence of assumption (3.6), the second follows from the definition of monomial valuations and the last one from Remark 3.5.

It remains to prove that a_j are integers. By assumption f is homogeneous with respect to $\text{Cl}(Y)$ -grading, and since ϕ_i are also homogeneous we may assume that $\phi_1^{k_1} \cdots \phi_n^{k_n}$ has the same degree as f . Indeed, images of summands $\alpha_{K'} \cdot w_1^{k'_1} \cdots w_n^{k'_n}$ mapped by κ to elements of different degrees must cancel, and removing them from \tilde{f} does not decrease $\tilde{\nu}_i(\tilde{f})$. But then $\frac{\phi_1^{k_1} \cdots \phi_n^{k_n}}{f} \in \mathbb{C}(V)^G$ and by Lemma 2.12

$$a_j = \frac{1}{r_j} \nu_j \left(\frac{\phi_1^{k_1} \cdots \phi_n^{k_n}}{f} \right) = \nu_{E_j} \left(\frac{\phi_1^{k_1} \cdots \phi_n^{k_n}}{f} \right) \in \mathbb{Z}.$$

□

PROOF OF THEOREM 3.8. By Proposition 2.9 (see also Remark 3.9) we have $\overline{\Theta}(\mathcal{R}(X)) \supset \mathcal{R}$, hence it suffices to show $\overline{\Theta}(\mathcal{R}(X)) \subset \mathcal{R}$.

We will use Proposition 3.13. Since $\overline{\Theta}$ is a graded ring homomorphism, it is sufficient to show that the images of homogeneous elements via $\overline{\Theta}$ are all in \mathcal{R} . Take $f = f_D \in \mathcal{R}(X)$. We have to show that $\overline{\Theta}(f) = f_{\varphi_* D} \cdot \prod_i t_i^{D \cdot C_i} \in \mathcal{R}$. We may write $D = \overline{\varphi_* D} + \sum_j a_j E_j$ and we obtain

$$f_{\varphi_* D} \cdot \prod_i t_i^{D \cdot C_i} = f_{\varphi_* D} \cdot \prod_i t_i^{\overline{\varphi_* D} \cdot C_i} \cdot \prod_j \left(\prod_i t_i^{E_j \cdot C_i} \right)^{a_j}.$$

This is an element of \mathcal{R} , because $f_{\varphi_* D} \cdot \prod_i t_i^{\overline{\varphi_* D} \cdot C_i} \in \mathcal{R}$ by Proposition 3.13 and the remaining factors are expressed in terms of the chosen generating set \mathcal{S} of \mathcal{R} . □

4. The symmetric group S_3

4.1. Generators of the Cox ring. In this section we investigate an action of $G = S_3$ on $V \simeq \mathbb{C}^4$ with coordinates (x_1, y_1, x_2, y_2) . We take the direct sum of two copies of the standard two-dimensional representation of S_3 , which preserves the symplectic form $\omega = dx_1 \wedge dy_2 + dy_1 \wedge dx_2$ on V . That is, we identify G with the matrix group generated by

$$\varepsilon = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon^{-1} & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon^{-1} \end{pmatrix} \quad T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where ϵ is a 3rd root of unity. Let ν be the monomial valuation $\mathbb{C}(V) \rightarrow \mathbb{Z}$ associated with symplectic reflection T . The following proposition summarizes basic properties of this representation; all of them are immediate.

PROPOSITION 4.1. *With the notation above:*

- Commutator subgroup $H = [G, G]$ is a cyclic group of order 3 generated by ε . In particular H does not contain any symplectic reflection.

- $\text{Ab}(G) = G/H = \langle TH \rangle \cong \mathbb{Z}_2$.
- Symplectic reflections in G are $\varepsilon^i T$ for $i = 0, 1, 2$. These elements correspond to transpositions in S_3 and as such generate G .
- All those symplectic reflections are conjugate. In particular, by the McKay correspondence, the symplectic resolution $\varphi: X \rightarrow Y = V/G$ has irreducible exceptional divisor E .
- The intersection matrix is the Cartan matrix of type A_1 . In other words, if C is a generic fiber of $\varphi|_X: E \rightarrow \varphi(E)$, then we have $E.C = -2$.
- $\nu|_{\mathbb{C}(V)^G} = 2\nu_E$ by Lemma 2.12.

eigenvalue generators

1	$\phi_1 = x_1 y_1, \quad \phi_2 = x_2 y_2, \quad \phi_3 = x_1 y_2 + x_2 y_1$ $\phi_4 = x_1^3 + y_1^3, \quad \phi_5 = x_2^3 + y_2^3, \quad \phi_6 = x_1^2 x_2 + y_1^2 y_2, \quad \phi_7 = x_1 x_2^2 + y_1 y_2^2$
-1	$\phi_8 = x_1 y_2 - x_2 y_1$ $\phi_9 = x_1^3 - y_1^3, \quad \phi_{10} = x_2^3 - y_2^3, \quad \phi_{11} = x_1^2 x_2 - y_1^2 y_2, \quad \phi_{12} = x_1 x_2^2 - y_1 y_2^2.$

TABLE 1. generators of $\mathbb{C}[V]^H$ that are eigenvectors of $\text{Ab}(G)$

PROPOSITION 4.2. *The elements ϕ_i in Table 1 are eigenvectors of the action of $\text{Ab}(G)$ which generate the ring of invariants $\mathcal{P} = \mathbb{C}[V]^H \subset \mathbb{C}[V]$.*

PROOF. We compute this generating set using Singular, [DGPS12]. First we find the invariants of the action of $[G, G]$. Then we split their linear span into smaller $\text{Ab}(G)$ -invariant subspaces on which the action of $\text{Ab}(G)$ is easy to diagonalize. \square

REMARK 4.3. We have $\nu(\phi_i) = 0$ if $T(\phi_i) = \phi$ and $\nu(\phi_i) = 1$ if $T(\phi_i) = -\phi_i$. Since in this case $\nu(\phi_i) = D_i.C$ by (3.10), then we have

- $\overline{\phi}_i = \phi_i$ if $T(\phi) = \phi$,
- $\overline{\phi}_i = \phi_i t$ if $T(\phi_i) = -\phi_i$.

PROPOSITION 4.4. *The image $\Theta(\mathcal{R}(X)) \subset \mathcal{P}[t^{\pm 1}]$ of the Cox ring of the symplectic resolution $X \rightarrow V/G$ is generated by the elements $\phi_1, \dots, \phi_7, \phi_8 t, \dots, \phi_{12} t$ and t^{-2} .*

PROOF. To use Theorem 3.8 we just have to show that condition 3.6 is satisfied. And this is the aim of the next section. \square

4.2. Lifting homogeneous elements. Here we give an argument for condition (3.6) in the case of considered representation of S_3 . In some sense this is the simplest possible case: there is only one conjugacy class of symplectic reflections, hence when choosing a lifting, we do not have to consider the interplay between different valuations $\tilde{\nu}_i$. However, the fact that there are three conjugate symplectic reflections leads to some complications which do not appear if the conjugacy classes have just two elements, as it happens for $\mathbb{Z}_2 \wr S_2$, cf. section 5.2.

Let $\kappa: \mathbb{C}[w_1, \dots, w_{12}] \rightarrow \mathcal{P}$ be the surjective ring homomorphism which assigns ϕ_i to w_i . Define a linear map $\tilde{T}: \mathbb{C}^{12} \rightarrow \mathbb{C}^{12}$ by setting

$$\tilde{T}(w_1, \dots, w_{12}) = (w_1, \dots, w_7, -w_8, \dots, -w_{12})$$

and take the corresponding valuation $\tilde{\nu}$, as in Definition 3.4.

We start with noticing that valuations ν and $\tilde{\nu}$ correspond to certain ideals generated by linear forms. Take the ideals of subspaces of \mathbb{C}^4 fixed by symplectic reflections in G :

$$\begin{aligned}\mathcal{K} &= I(\ker(T - \text{id})) = (x_1 - y_1, x_2 - y_2), \\ \mathcal{K}' &= I(\ker(\varepsilon T - \text{id})) = (x_1 - \varepsilon y_1, x_2 - \varepsilon y_2), \\ \mathcal{K}'' &= I(\ker(\varepsilon^{-1}T - \text{id})) = (x_1 - \varepsilon^{-1}y_1, x_2 - \varepsilon^{-1}y_2).\end{aligned}$$

Also, set $\mathcal{J} = (w_8, \dots, w_{12}) \triangleleft \mathbb{C}[w_1, \dots, w_{12}]$.

LEMMA 4.5. *The valuation ν is the valuation of the ideal \mathcal{K} , that is $\nu(f) = d$ is equivalent to $f \in \mathcal{K}^d$ and $f \notin \mathcal{K}^{d+1}$. The analogous statement is true for the valuation $\tilde{\nu}$ and the ideal \mathcal{J} .*

PROOF. After a linear change of coordinates $T = \text{diag}(-1, -1, 1, 1)$ we may write $\mathcal{K} = (x_1, y_1)$. Then both statements $\nu(f) \geq d$ and $f \in \mathcal{K}^d$ are equivalent to the fact that each monomial of f is of degree at least d with respect to x_1, y_1 . Similar argument proves the statement concerning $\tilde{\nu}$. \square

Recall that $\mathcal{P} \simeq \mathcal{R}(Y)$ is graded by $\text{Cl}(Y) = G^\vee \simeq \text{Ab}(G)^\vee$. We will consider elements homogeneous with respect to this grading.

LEMMA 4.6. *If $f \in \mathcal{P}$ is homogeneous with respect to $\text{Cl}(Y)$ -grading and $f \in \mathcal{K}^d$ then $f \in \mathcal{K}^d \cap (\mathcal{K}')^d \cap (\mathcal{K}'')^d$.*

PROOF. Symplectic reflections $T, \varepsilon T$ are conjugate, i.e. $\varepsilon T = gTg^{-1}$ for some $g \in G$. Thus $f \in \mathcal{K}^d = I(\ker(T - \text{id}))^d$ implies that $gf \in I(\ker(g(T - \text{id})g^{-1}))^d = (\mathcal{K}')^d$. Since f is homogeneous, G acts on f by multiplication by some character $\chi \in G^\vee$. In particular, $gf = \chi(g)f$, where $\chi(g) \in \mathbb{C}^*$. Hence $f \in (\mathcal{K}')^d$, and in the same way $f \in (\mathcal{K}'')^d$. \square

REMARK 4.7. We are going to prove condition (3.6), which in this case can be stated as follows: for each homogeneous $f \in \mathcal{P}$ there exists $\tilde{f} \in \mathbb{C}[w_1, \dots, w_{12}]$ such that $\kappa(\tilde{f}) = f$ and $\nu(f) = \tilde{\nu}(\tilde{f})$.

Let \mathcal{I} be the kernel of κ . By Lemma 4.6, in terms of corresponding ideals this condition can be expressed as

$$(4.8) \quad \kappa^{-1}(\mathcal{K}^d \cap (\mathcal{K}')^d \cap (\mathcal{K}'')^d) \subseteq \mathcal{J}^d + \mathcal{I}.$$

The remaining part of this section is a proof of (4.8). We start from a technical argument that in this case instead of considering intersection of powers of ideals we may look at powers of intersections.

LEMMA 4.9. *For $d \in \mathbb{N}$ we have $\mathcal{K}^d \cap (\mathcal{K}')^d \cap (\mathcal{K}'')^d = (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d$.*

PROOF. After a linear change of coordinates we may assume that two of considered ideals are monomial: $\mathcal{K} = (z_1, z_2)$, $\mathcal{K}' = (z_3, z_4)$ and $\mathcal{K}'' = (z_1 + z_3, z_2 + z_4)$ in $\mathbb{C}[z_1, z_2, z_3, z_4]$. Also, one checks directly (e.g. in Macaulay2, [GS13]) that

$$\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'' = (z_1 z_4 - z_2 z_3) + \mathcal{K} \cdot \mathcal{K}' \cdot \mathcal{K}''.$$

Obviously the right-hand side is contained in the left-hand side. We prove the other inclusion. The proof goes by induction on d . For $d = 1$ the result is clear. We

assume that $\mathcal{K}^{d-1} \cap (\mathcal{K}')^{d-1} \cap (\mathcal{K}'')^{d-1} = (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^{d-1}$ and prove the inclusion in

$$\mathcal{K}^d \cap (\mathcal{K}')^d \cap (\mathcal{K}'')^d \subseteq (\mathcal{K}^{d-1} \cap (\mathcal{K}')^{d-1} \cap (\mathcal{K}'')^{d-1}) \cdot (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'') = (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d.$$

Note that $(\mathcal{K}' \cdot \mathcal{K}'')^d = (\mathcal{K}')^d \cap (\mathcal{K}'')^d$, since these ideals are generated by linearly independent forms (i.e. after a linear change of coordinates they become monomial, generated in independent variables). Hence the left-hand side can be written as $\mathcal{K}^d \cap (\mathcal{K}' \cdot \mathcal{K}'')^d$. Consider an element α of the left-hand side. We express it in terms of generators of $(\mathcal{K}' \cdot \mathcal{K}'')^d$ and then explore the information that it belongs to \mathcal{K}^d :

$$\alpha = \sum_{i \in I} C_i (z_1 + z_3)^{a_i} (z_2 + z_4)^{d-a_i} z_3^{b_i} z_4^{d-b_i} w_i,$$

where w_i are monomials, C_i are constants and I is a set of indices. Assume that α is homogeneous (with respect to standard degree) – we can work in each degree separately.

Now we show by induction that α can be written as a sum of element divisible by $z_1 z_4 - z_2 z_3$ and an element of $(\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d$. We use the (descending) induction on the minimal pair (m_i, n_i) such that m_i is the degree of w_i in z_1 and z_2 (i.e. sum of the degree in z_1 and in z_2) and n_i is the degree of w_i in z_3 , ordered lexicographically. We may assume that α contains components for which $(m_i, n_i) < (d, 0)$ (i.e. w_i is not divisible by any $z_1^c z_2^{d-c}$), since otherwise $\alpha \in (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d$.

Let $J \subseteq I$ be the set of indices with minimal (m_i, n_i) , equal to (m, n) . Look at

$$\alpha' = z_1^n z_2^{m-n} \sum_{i \in J} C_i (z_1 + z_3)^{a_i} (z_2 + z_4)^{d-a_i} z_3^{b_i} z_4^{d-b_i} z_3^{e_i} z_4^{f-e_i},$$

where f is the same for all terms because of the homogeneity. It is enough to show that $\alpha' \in (z_1 z_4 - z_2 z_3)$, the rest follows from the induction hypothesis for $\alpha - \alpha'$.

We do another (decreasing) induction on the minimal value of a_i for $i \in J$. Let a_0 be this minimal value, assume first that $a_0 < d$. Then for every component of the sum above with $a_i = a_0$ we modify α' to $\alpha' - (S_0 - S_1)$, where S_0 is the considered component (with exponents b_0 and e_0) and

$$S_1 = C_i (z_1 + z_3)^{a_0+1} (z_2 + z_4)^{d-a_0-1} z_3^{b_0+e_0-1} z_4^{d+f+1-b_0-e_0}.$$

We may assume that $b_0 + e_0 > 0$, because otherwise we have a monomial in α' which is not in \mathcal{K}^d , but cannot cancel with any other: this is $c_0 z_1^n z_2^{m-n} z_3^{a_0} z_4^{2d+f-a_0}$ (obtained by choosing always z_3 and z_4 from brackets).

Note that every time we subtract from α' a polynomial divisible by $z_1 z_4 - z_2 z_3$:

$$S_0 - S_1 = C_i (z_1 + z_3)^{a_0} (z_2 + z_4)^{d-a_0-1} z_3^{b_0+e_0-1} z_4^{d+f-b_0-e_0} ((z_2 + z_4) z_3 - (z_1 + z_3) z_4).$$

Also, since S_1 is divisible by $(z_1 + z_3)^{a_0+1}$, after all these modifications we have increased the minimal value of a_i .

Finally, if $a_0 = d$ then the monomial obtained by choosing z_3 in every bracket in the corresponding sum element (remember that there is no $(z_2 + z_4)$ in the component where $b_0 + e_0$ is maximal) is not in \mathcal{K}^d and it cannot delete with any other monomial.

Summing up, we can write α' , hence also α , in the form $(z_1 z_4 - z_2 z_3)\beta + \gamma$, where $\gamma \in (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d$. Computing monomial valuations corresponding to $\mathcal{K}, \mathcal{K}', \mathcal{K}''$ we get $\beta \in \mathcal{K}^{d-1}$, $\beta \in (\mathcal{K}')^{d-1}$, $\beta \in (\mathcal{K}'')^{d-1}$, which ends the outer induction.

□

LEMMA 4.10. *The intersection $\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}''$ is generated by the following polynomials*

$$y_1x_2 - x_1y_2, \ x_2^3 - y_2^3, \ x_1x_2^2 - y_1y_2^2, \ x_1^2x_2 - y_1^2y_2, \ x_1^3 - y_1^3.$$

which are elements of \mathcal{P} . Moreover, we have

$$(\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d \cap \mathcal{P} = (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'' \cap \mathcal{P})^d.$$

PROOF. The first part is a direct computation, e.g. in Macaulay2. To prove the second part note that \mathcal{P} , as the ring of invariants of $[G, G]$, is generated by monomials of the form $x_1^{\alpha_1} y_1^{\beta_1} x_2^{\alpha_2} y_2^{\beta_2}$, where $\alpha_1 + \alpha_2 \equiv \beta_1 + \beta_2 \pmod{3}$. Only one inclusion needs to be proved: take $f \in (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d \cap \mathcal{P}$, we show that it belongs to the right-hand side. Since $f \in (\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d$, we may write

$$f = \sum_{i_1 + \dots + i_5 = d} p_1^{i_1} \cdots p_5^{i_5} \alpha_{i_1, \dots, i_5},$$

where p_i are generators of $\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}''$ listed above and $\alpha_{i_1, \dots, i_5} \in \mathbb{C}[x_1, y_1, x_2, y_2]$. And since $f \in \mathcal{P}$, taking the average over $[G, G]$ we get

$$f = \frac{1}{|[G, G]|} \sum_{g \in [G, G]} gf = \sum_{i_1 + \dots + i_5 = d} p_1^{i_1} \cdots p_5^{i_5} \bar{\alpha}_{i_1, \dots, i_5},$$

where $\bar{\alpha}_{i_1, \dots, i_5} = \frac{1}{|[G, G]|} \sum_{g \in G} g \alpha_{i_1, \dots, i_5} \in \mathcal{P}$. And each element of this sum already belongs to $(\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'' \cap \mathcal{P})^d$. □

Recall that $\mathcal{I} = \ker \kappa$, take the pull-backs of considered ideals through κ : $\tilde{\mathcal{K}} = \kappa^{-1}(\mathcal{K})$, $\tilde{\mathcal{K}}' = \kappa^{-1}(\mathcal{K}')$, $\tilde{\mathcal{K}}'' = \kappa^{-1}(\mathcal{K}'')$.

COROLLARY 4.11. *We have*

$$\kappa^{-1}((\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d) = (\tilde{\mathcal{K}} \cap \tilde{\mathcal{K}}' \cap \tilde{\mathcal{K}}'')^d + \mathcal{I}.$$

PROOF. Recall that κ is an epimorphism onto \mathcal{P} . In particular, if A, A' are ideals of \mathcal{P} then $\kappa^{-1}(AA') = \kappa^{-1}(A)\kappa^{-1}(A') + \mathcal{I}$. Using this fact together with Lemma 4.10 we obtain

$$\begin{aligned} \kappa^{-1}((\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d) &= \kappa^{-1}((\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'')^d \cap \mathcal{P}) = \\ &= \kappa^{-1}((\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'' \cap \mathcal{P})^d) = (\kappa^{-1}(\mathcal{K} \cap \mathcal{K}' \cap \mathcal{K}'' \cap \mathcal{P}))^d + \mathcal{I} = (\tilde{\mathcal{K}} \cap \tilde{\mathcal{K}}' \cap \tilde{\mathcal{K}}'')^d + \mathcal{I}. \end{aligned}$$

□

The last observation can be verified by a direct computation in Macaulay2.

LEMMA 4.12. $\tilde{\mathcal{K}} \cap \tilde{\mathcal{K}}' \cap \tilde{\mathcal{K}}'' = \mathcal{J} + \mathcal{I}$.

Summing up, condition (4.8) follows by Lemma 4.9, Corollary 4.11 and Lemma 4.12. This finishes the proof of Proposition 4.4.

5. Wreath product $\mathbb{Z}_2 \wr S_2$

5.1. Representation. In this part we consider the symplectic action of the wreath product $G \simeq \mathbb{Z}_2 \wr S_2$ on $V = \mathbb{C}^4$ with coordinates (x_1, x_2, x_3, x_4) and symplectic form $dx_1 \wedge dx_3 + dx_2 \wedge dx_4$. The action is given by the embedding $\mathbb{Z}_2 \subset \mathrm{SL}_2(\mathbb{C})$ and by permutation of factors $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. More precisely, G is generated by matrices

$$T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

REMARK 5.1. This is a subgroup of the 32-element group investigated in [DBW14]. Our notation for symplectic reflections is compatible with the one used therein.

Below we list basic properties of G , the arguments are straightforward. Let $\nu_0, \nu_2 : \mathbb{C}(V)^{[G,G]} \rightarrow \mathbb{Z}$ denote monomial valuations corresponding to symplectic reflections T_0, T_2 respectively.

PROPOSITION 5.2. *With notation as above:*

- G is isomorphic to the dihedral group D_8 of order 8, the isomorphism is given by identifying elements $\pm T_0, \pm T_2$ with reflections and $T_0 T_2$ with rotation.
- The commutator subgroup $H = [G, G] \simeq \mathbb{Z}_2$ is generated by matrix $-\mathrm{id}$. In particular, H does not contain any symplectic reflection.
- $\mathrm{Ab}(G) = G/H = \langle T_0 H, T_2 H \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$
- Matrices $\pm T_0$ and $\pm T_2$ are all the symplectic reflections in G . Their conjugacy classes are $\{\pm T_0\}$, $\{\pm T_2\}$. In particular, by the McKay correspondence the exceptional divisor of a symplectic resolution $\varphi : X \rightarrow Y = V/G$ has two irreducible components E_0, E_2 corresponding to these conjugacy classes.
- The intersection matrix $(E_i \cdot C_j)_{i,j}$ is a direct sum of two A_1 -type Cartan matrices, i.e.

$$(E_i \cdot C_j)_{i,j} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

- $\nu_i|_{\mathbb{C}(V)^H} = 2\nu_{E_i}$, $i = 0, 2$ by Lemma 2.12.

PROPOSITION 5.3. *The elements ϕ_{ij} in Table 2 are eigenvectors of the action of $\mathrm{Ab}(G)$, which generate the ring of invariants $\mathbb{C}[V]^H \subset \mathbb{C}[V]$.*

PROOF. See [DBW14, Lem. 3.12]. These polynomials are eigenvectors of the action of the abelianization of the 32-element group investigated therein. Since G is its subgroup and commutator subgroups are equal, they are also eigenvectors for $\mathrm{Ab}(G)$. \square

generators	eigenvalues	
	T_0	T_2
$\phi_{13} = -x_1^2 - x_2^2 + x_3^2 + x_4^2$	1	1
$\phi_{14} = \sqrt{-1}(x_1^2 + x_2^2 + x_3^2 + x_4^2)$	1	1
$\phi_{34} = 2(x_1x_3 + x_2x_4)$	1	1
$\phi_{01} = -2(x_1x_4 + x_2x_3)$	-1	1
$\phi_{03} = 2\sqrt{-1}(x_1x_2 + x_3x_4)$	-1	1
$\phi_{04} = 2(-x_1x_2 + x_3x_4)$	-1	1
$\phi_{12} = 2(x_1x_3 - x_2x_4)$	1	-1
$\phi_{23} = \sqrt{-1}(-x_1^2 + x_2^2 - x_3^2 + x_4^2)$	1	-1
$\phi_{24} = x_1^2 - x_2^2 - x_3^2 + x_4^2$	1	-1
$\phi_{02} = 2\sqrt{-1}(x_1x_4 - x_2x_3)$	-1	-1

TABLE 2. Generators of $\mathbb{C}[V]^H$ which are eigenvectors of $\text{Ab}(G)$ (cf. [DBW14, 3.13]).

REMARK 5.4. Using (3.10) we obtain an explicit description of generators $\bar{\phi}_{ij}$ of the Cox ring of symplectic resolutions of V/G , proposed in Theorem 3.8:

$$(5.5) \quad \bar{\phi}_{ij} = \begin{cases} \phi_{ij}t_0t_2 & \text{if } \{0, 2\} = \{i, j\}, \\ \phi_{ij}t_0 & \text{if } \{0, 2\} \cap \{i, j\} = \{0\}, \\ \phi_{ij}t_2 & \text{if } \{0, 2\} \cap \{i, j\} = \{2\}, \\ \phi_{ij}, & \text{if } \{0, 2\} \cap \{i, j\} = \emptyset. \end{cases}$$

To simplify the notation we set $\phi_{21} = \phi_{12}$.

PROPOSITION 5.6. *Elements $t_0^{-2}, t_2^{-2}, \phi_{02}t_0t_2, \phi_{01}t_0, \phi_{2j}t_2, \phi_{ij}$, for $i, j = 1, 3, 4$, $i < j$ generate the image $\Theta(\mathcal{R}(X)) \subset \mathcal{P}[t_0^{\pm 1}, t_2^{\pm 1}]$ of the Cox ring of symplectic resolutions $X \rightarrow V/G$.*

We will denote by \mathcal{R} the ring generated by elements listed in Proposition 5.6.

In section 5.2 we give a part of the proof of Proposition 5.6. A complete argument, based on different methods than these introduced here, will be presented in [DB15]. Next, in sections 5.3-5.6, we investigate the geometry of certain GIT quotients of $\text{Spec } \mathcal{R}$: we prove their smoothness and provide an explicit description of the central fiber of a resolution.

5.2. Lifting homogeneous elements. Here we prove a weaker version of condition (3.6) for $G = \mathbb{Z}_2 \wr S_2$. In Lemma 5.13 we show that it is satisfied for each valuation ν_0, ν_2 separately. The proof goes along the same lines as the one for S_3 in section 4.2. However, the argument is simpler since in $\mathbb{Z}_2 \wr S_2$ each conjugacy class of symplectic reflections has only two elements.

Let $\kappa: \mathbb{C}[w_{ij}: 0 \leq i < j \leq 4] \rightarrow \mathcal{P}$ be the surjective ring homomorphism which sends w_{ij} to the eigenvectors ϕ_{ij} from Table 2.

Define linear maps $\tilde{T}_0, \tilde{T}_2: \mathbb{C}^{10} \rightarrow \mathbb{C}^{10}$:

$$\tilde{T}_0(e_{ij}) = \begin{cases} e_{ij} & \text{if } 0 \notin \{i, j\} \\ -e_{ij} & \text{if } 0 \in \{i, j\} \end{cases} \quad \tilde{T}_2(e_{ij}) = \begin{cases} e_{ij} & \text{if } 2 \notin \{i, j\} \\ -e_{ij} & \text{if } 2 \in \{i, j\} \end{cases}$$

where $\{e_{ij} : 0 \leq i < j \leq 4\}$ is a basis of \mathbb{C}^{10} dual to the set of variables w_{ij} . By $\tilde{\nu}_0, \tilde{\nu}_2$ we denote corresponding monomial valuations.

Note that considered monomial valuations correspond to ideals generated by linear forms. For each conjugacy class of symplectic reflections in G and for each representative we write the ideal of the subspace of \mathbb{C}^4 fixed by this reflection:

$$\begin{aligned}\mathcal{K}_0 &:= I(\ker(\text{id} - T_0)) = (x_1, x_3), \\ \mathcal{K}'_0 &:= I(\ker(\text{id} + T_0)) = (x_2, x_4), \\ \mathcal{K}_2 &:= I(\ker(\text{id} - T_2)) = (x_1 - x_2, x_3 - x_4), \\ \mathcal{K}'_2 &:= I(\ker(\text{id} + T_2)) = (x_1 + x_2, x_3 + x_4).\end{aligned}$$

Also, define ideals $\mathcal{J}_0, \mathcal{J}_2 \triangleleft \mathbb{C}[w_{ij} : 0 \leq i < j \leq 4]$ by setting $\mathcal{J}_0 = (w_{01}, w_{02}, w_{03}, w_{04})$ and $\mathcal{J}_2 = (w_{02}, w_{12}, w_{23}, w_{24})$. The following properties can be proved in the same way as Lemmas 4.5 and 4.6.

LEMMA 5.7. *The valuation ν_i is the valuation of ideal \mathcal{K}_i , that is $\nu_i(f) = d$ is equivalent to $f \in \mathcal{K}_i^d$ and $f \notin \mathcal{K}_i^{d+1}$ for $i = 0, 2$. The analogous statement is true for valuations $\tilde{\nu}_i$ and ideals \mathcal{J}_i .*

LEMMA 5.8. *If $f \in \mathcal{P}$ is homogeneous with respect to the $\text{Cl}(Y)$ -grading and $f \in \mathcal{K}_i^d$, i.e. $\nu_i(f) \geq d$, then $f \in \mathcal{K}_i^d \cap (\mathcal{K}'_i)^d$.*

Next we investigate the intersections $\mathcal{K}_i^d \cap (\mathcal{K}'_i)^d$ and their preimages under κ .

LEMMA 5.9. *For $d \in \mathbb{N}$ and $i = 0, 2$ we have $(\mathcal{K}_i \cap \mathcal{K}'_i)^d = \mathcal{K}_i^d \cap (\mathcal{K}'_i)^d$.*

PROOF. It is sufficient to notice that after a suitable change of coordinates \mathcal{K}_i and \mathcal{K}'_i are monomial ideals generated by independent variables. \square

LEMMA 5.10. *We have $\mathcal{K}_0 \cap \mathcal{K}'_0 = (x_1x_2, x_1x_4, x_2x_3, x_3x_4)$ and $\mathcal{K}_2 \cap \mathcal{K}'_2 = ((x_1 - x_2)(x_1 + x_2), (x_3 - x_4)(x_3 + x_4), (x_1 - x_2)(x_3 + x_4), (x_3 - x_4)(x_1 + x_2))$. In particular, both ideals $\mathcal{K}_i \cap \mathcal{K}'_i$, $i = 0, 2$ are generated by elements from \mathcal{P} . Consequently*

$$(\mathcal{K}_i \cap \mathcal{K}'_i)^d \cap \mathcal{P} = (\mathcal{K}_i \cap \mathcal{K}'_i \cap \mathcal{P})^d.$$

PROOF. The description of $\mathcal{K}_0 \cap \mathcal{K}'_0$ is obvious and for $\mathcal{K}_2 \cap \mathcal{K}'_2$ it follows easily because after a change of coordinates \mathcal{K}_2 and \mathcal{K}'_2 become monomial ideals generated in independent variables. The proof of the second part is based on the fact that \mathcal{P} is the ring of invariants of $[G, G]$, as in Lemma 4.10. \square

Define $\tilde{\mathcal{K}}_i = \kappa^{-1}(\mathcal{K}_i)$, $\tilde{\mathcal{K}}'_i = \kappa^{-1}(\mathcal{K}'_i)$ for $i = 0, 2$, and let $\mathcal{I} = \ker \kappa$.

COROLLARY 5.11. *For $i = 0, 2$ Lemma 5.10 gives*

$$\kappa^{-1}((\mathcal{K}_i \cap \mathcal{K}'_i)^d) = (\tilde{\mathcal{K}}_i \cap \tilde{\mathcal{K}}'_i)^d + \mathcal{I}.$$

Finally, the following lemma is proved by a computation in Macaulay2, [GS13].

LEMMA 5.12. *For $i = 0, 2$ we have $\mathcal{J}_i + \mathcal{I} = \tilde{\mathcal{K}}_i \cap \tilde{\mathcal{K}}'_i$.*

Summing up, we may lift homogeneous elements of \mathcal{P} compatibly with one valuation.

LEMMA 5.13. *Let $\tilde{\nu}_i$ be the monomial valuation of \tilde{T}_i , $i = 0, 2$. Then, for $i = 0, 2$ and every homogeneous $f \in \mathcal{P}$ there exists $\tilde{f} \in \mathbb{C}[w_{ij} : 0 \leq i < j \leq 4]$ such that $\kappa(\tilde{f}) = f$ and $\nu_i(f) = \tilde{\nu}_i(\tilde{f})$.*

PROOF. Assume that $f \in \mathcal{K}_i^d$. Then by Lemma 5.8 $f \in \mathcal{K}_i^d \cap \mathcal{K}_i'^d$, and by Corollary 5.11 and Lemma 5.12 we have

$$\kappa^{-1}(\mathcal{K}_i^d \cap (\mathcal{K}_i')^d) = \kappa^{-1}((\mathcal{K}_i \cap \mathcal{K}_i')^d) = (\tilde{\mathcal{K}}_i \cap \tilde{\mathcal{K}}_i')^d + \mathcal{I} = (\mathcal{J}_i + \mathcal{I})^d + \mathcal{I} = \mathcal{J}_i^d + \mathcal{I}.$$

□

5.3. The ideal of an embedding of $\text{Spec } \mathcal{R}$ in an affine space. We are now going to describe the ideal of an embedding $\text{Spec } \mathcal{R} \hookrightarrow \mathbb{C}^{12}$ and the relation with an analogous embedding constructed in case of the group of order 32 considered in [DBW14].

To embed $\text{Spec } \mathcal{R}$ in \mathbb{C}^{12} we consider a surjective homomorphism

$$\Psi: \mathbb{C}[w_{ij}, u_0, u_2 : 0 \leq i < j \leq 4] \rightarrow \mathcal{R}$$

which sends w_{ij} to $\bar{\phi}_{ij}$, listed in Remark 5.4, and u_k to t_k^{-2} for $k = 0, 2$.

If \mathcal{R}' is the subring of $\mathcal{P}[t_0^{\pm 1}, \dots, t_4^{\pm 1}]$ generated by $\phi_{ij}t_it_j$, $0 \leq i < j \leq 4$ and t_k^{-2} , $k = 0, \dots, 4$ then as in [DBW14, Prop. 3.17] we may consider a closed embedding $\text{Spec } \mathcal{R}' \rightarrow \mathbb{C}^{15}$. It is given by a surjective ring homomorphism

$$\Phi: \mathbb{C}[w_{ij}, u_k : k = 0, \dots, 4, 0 \leq i < j \leq 4] \rightarrow \mathcal{R}',$$

which sends w_{ij} to $\phi_{ij}t_it_j$ and u_k to t_k^{-2} .

Note that Ψ is the composition of Φ with the map $\mathcal{P}[t_0^{\pm 1}, \dots, t_4^{\pm 1}] \rightarrow \mathcal{P}[t_0^{\pm 1}, t_2^{\pm 1}]$ such that t_k is mapped to 1 for $k = 1, 3, 4$ and to t_k for $k = 0, 2$. Thus we obtain a commutative diagram of closed embeddings

$$(5.14) \quad \begin{array}{ccc} \text{Spec } \mathcal{R} & \longrightarrow & \mathbb{C}^{12} \\ \downarrow & & \downarrow \\ \text{Spec } \mathcal{R}' & \longrightarrow & \mathbb{C}^{15} \end{array}$$

where \mathbb{C}^{12} is an affine subspace of \mathbb{C}^{15} given by $u_1 = u_3 = u_4 = 1$.

PROPOSITION 5.15. *The kernel of Ψ , i.e. the ideal of the embedding $\text{Spec } \mathcal{R} \hookrightarrow \mathbb{C}^{12}$, is generated by the following polynomials:*

$$\begin{array}{ll} w_{14}w_{23} + w_{13}w_{24} - w_{12}w_{34} & w_{04}w_{23} - w_{03}w_{24} - w_{02}w_{34} \\ w_{04}w_{13} + w_{03}w_{14} - w_{01}w_{34} & w_{04}w_{12} - w_{02}w_{14} - w_{01}w_{24} \\ w_{03}w_{12} + w_{02}w_{13} - w_{01}w_{23} & \\ w_{02}w_{12}u_2 - w_{03}w_{13} + w_{04}w_{14} & w_{01}w_{14} - w_{02}w_{24}u_2 + w_{03}w_{34} \\ w_{01}w_{13} + w_{02}w_{23}u_2 + w_{04}w_{34} & w_{01}w_{12} + w_{03}w_{23} + w_{04}w_{24} \\ w_{03}w_{04}u_0 - w_{13}w_{14} + w_{23}w_{24}u_2 & w_{02}w_{04}u_0 + w_{12}w_{14} + w_{23}w_{34} \\ w_{01}w_{04}u_0 + w_{12}w_{24}u_2 + w_{13}w_{34} & w_{02}w_{03}u_0 - w_{12}w_{13} - w_{24}w_{34} \\ w_{01}w_{03}u_0 + w_{12}w_{23}u_2 + w_{14}w_{34} & w_{01}w_{02}u_0 + w_{13}w_{23} - w_{14}w_{24} \\ w_{02}^2u_0 + w_{12}^2 + w_{23}^2 + w_{24}^2 & w_{03}^2u_0 + w_{13}^2 + w_{23}^2u_2 + w_{34}^2 \\ w_{01}^2 + w_{02}^2u_2 + w_{03}^2 + w_{04}^2 & w_{04}^2u_0 + w_{14}^2 + w_{24}^2u_2 + w_{34}^2 \\ w_{01}^2u_0 + w_{12}^2u_2 + w_{13}^2 + w_{14}^2 & \end{array}$$

Note that according to [DBW14, Prop. 3.17] these polynomials are exactly the generators of $\ker \Phi$ after substituting $u_i \mapsto 1$ for $i = 1, 3, 4$. In particular, $\text{Spec } \mathcal{R} = \mathbb{C}^{12} \cap \text{Spec } \mathcal{R}'$.

PROOF. This can be computed e.g. in Macaulay2, [GS13]. Another way of obtaining the generators of the ideal of $\text{Spec } \mathcal{R}$ is to observe that the ideal of the closed embedding $\text{Spec } \mathcal{R} \hookrightarrow \text{Spec } \mathcal{R}'$ is equal to $(t_1^{-2} - 1, t_3^{-2} - 1, t_4^{-2} - 1)$ and use the generators of the ideal of $\text{Spec } \mathcal{R}'$ listed in [DBW14, Prop. 3.17]. \square

5.4. GIT quotients of $\text{Spec } \mathcal{R}$ – linearization and stability. In the following sections we are going to show that any crepant resolution $X \rightarrow V/G$ is a GIT quotient of $\text{Spec } \mathcal{R}$ by an action of a two-dimensional torus $\mathbb{T}_{\text{Cl}(X)}$, and to describe the central fiber of such a resolution. We start from investigating appropriate linearizations of this action.

Consider the two-dimensional Picard torus $\mathbb{T} = \mathbb{T}_{\text{Cl}(X)}$ of X , which we identify with a subtorus of $\mathbb{T}' \cong (\mathbb{C}^*)^5$ given by $t_1 = t_3 = t_4 = 1$. We have a natural action of \mathbb{T} on $\text{Spec } \mathcal{R}$ induced by the inclusion $\mathcal{R} \subset \Theta(\mathcal{R}(X)) \subset \mathcal{P}[t_0^{\pm 1}, t_2^{\pm 1}]$.

More explicitly, $\mathbb{T} \cong \{(t_0, t_2) : t_0, t_2 \in \mathbb{C}^*\}$ acts trivially on w_{13}, w_{14}, w_{34} , by character t_0 on w_{01}, w_{03}, w_{04} , by character t_2 on w_{12}, w_{23}, w_{24} , by character $t_0 t_2$ on w_{02} , by character t_0^{-2} on u_0 and by character t_2^{-2} on u_2 .

NOTATION 5.16. By χ we denote the character $\chi^{(2,1)}$ of \mathbb{T} .

Our goal is to show the results analogous to [DBW14, Prop. 4.12, 4.14] for the \mathbb{T} -action on $\text{Spec } \mathcal{R}$ linearized by χ . In the present section we investigate the (semi)stability for this linearization. The subsequent one explains the smoothness of the geometric quotient of the stable locus. Note that by symmetry the same will be true then for $\chi^{(1,2)}$.

We describe explicitly the χ -semistable locus of $\text{Spec } \mathcal{R}$, the proof is straightforward.

LEMMA 5.17. *The set of χ -semistable points of \mathbb{C}^{12} is equal to*

$$(\mathbb{C}^{12})_{\chi}^{ss} = \bigcup_{i=1,3,4} \{w_{02}w_{0i} \neq 0\} \cup \bigcup_{i,j=1,3,4} \{w_{0i}w_{2j} \neq 0\},$$

where we denote $w_{21} := w_{12}$.

The next observation may be verified directly, using the above characterization of χ -semistable points.

PROPOSITION 5.18. *Every χ -semistable point of $\text{Spec } \mathcal{R}$ is χ -stable and has trivial isotropy group.*

5.5. The smoothness of GIT quotients. We will now show that a crepant resolution X can be reconstructed as a GIT quotient of $\text{Spec } \mathcal{R}$. To prove the smoothness of the quotient we follow the reasoning from the proof of an analogous result [DBW14, Prop. 4.14]. However, our case is simpler, it requires less computations. The main step is showing the smoothness of χ -stable locus of $\text{Spec } \mathcal{R}$. Then the smoothness of its geometric quotient follows from the facts that \mathbb{T} acts freely on this set.

PROPOSITION 5.19. *$\text{Spec } \mathcal{R}$ is smooth in every χ -stable point.*

PROOF. We use the Jacobian matrix M of the generating set of the ideal $\ker \Psi$ of $\text{Spec } \mathcal{R} \subset \mathbb{C}^{12}$ listed in Proposition 5.15. Its rows correspond to partial derivatives with respect to variables in the following order:

$$w_{01}, w_{02}, w_{03}, w_{04}, w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}, u_0, u_2.$$

As in the proof of [DBW14, Prop. 4.14] we look for a 6×6 minors of M which are monomials. Such minors give a vanishing of coordinates in singular points. We use the description of the set of stable points from Lemma 5.17) to conclude that points with certain sets of vanishing coordinates cannot be stable.

We use the following notation: $\det(i_1, \dots, i_6 | j_1, \dots, j_6)$ is the 6×6 minor of Jacobian matrix M of rows $i_1, \dots, i_6 \in \{1, \dots, 12\}$ and columns $j_1, \dots, j_6 \in \{1, \dots, 20\}$. By $\text{Mon}(x_{i_1}, \dots, x_{i_k})$ we denote the set of monomials in variables x_{i_1}, \dots, x_{i_k} .

We start by observing that $\det(4, 6, 7, 9, 10, 11 | 2, 5, 6, 9, 16, 18) \in \text{Mon}(w_{02}, w_{04})$ and also $\det(5, 6, 7, 10, 11, 12 | 2, 4, 5, 9, 16, 18) \in \text{Mon}(w_{01}, w_{02})$. Now if $w_{02} \neq 0$ then each singular point of $\text{Spec } \mathcal{R}$ satisfies $w_{01} = w_{04} = 0$, and substituting this zeroes to M we get $\det(6, 7, 8, 10, 11, 12 | 2, 3, 5, 9, 16, 18) \in \text{Mon}(w_{02}, w_{03})$. Given that $w_{02} \neq 0$ we get also $w_{03} = 0$ at each singular point. Together with $w_{01} = w_{04} = 0$ this implies that singular points satisfying $w_{02} \neq 0$ are not χ -stable.

It remains to consider singular points satisfying $w_{02} = 0$. Substituting this to M we get $\det(1, 6, 7, 8, 9, 11 | 4, 5, 7, 8, 18, 20) \in \text{Mon}(w_{01})$, hence $w_{01} = 0$. Substituting to M again, we have $\det(4, 5, 7, 9, 10, 11 | 4, 6, 8, 9, 18, 19) \in \text{Mon}(w_{04})$ and $\det(3, 5, 7, 9, 10, 11 | 2, 3, 5, 7, 17, 18) \in \text{Mon}(w_{03})$. Hence $w_{03} = w_{04} = 0$. Together with $w_{01} = 0$ this, as before, implies that singular points satisfying $w_{02} = 0$ are not χ -stable. \square

Following [DBW14, Prop. 3.16], we relate GIT quotients of $\text{Spec } \mathcal{R}$ to the singular space V/G .

LEMMA 5.20. *The homomorphism $\mathcal{R} \rightarrow \mathbb{C}[V]^{[G, G]}$ induces an isomorphism $\mathcal{R}^{\mathbb{T}} \cong \mathbb{C}[V]^G$.*

PROOF. Note that the group $\text{Ab}(G)$ can be presented as a subgroup of \mathbb{T} by sending the class of T_0 to $(-1, 1)$ and T_2 to $(1, -1)$. This makes the homomorphism $\mathcal{R} \rightarrow \mathbb{C}[V]^{[G, G]}$ an $\text{Ab}(G)$ -equivariant homomorphism, hence the image of $\mathcal{R}^{\mathbb{T}}$ is contained in $\mathbb{C}[V]^G$. It is injective on $\mathcal{R}^{\mathbb{T}}$, since $\mathcal{R}^{\mathbb{T}}$ has trivial intersection with its kernel $(t_0 - 1, t_2 - 1) \cap \mathcal{R}$. To prove it is onto, note that a $[G, G]$ -invariant element $\prod \phi_{ij}^{a_{ij}}$ is in $\mathbb{C}[V]^G$ if and only if $a_{01} + a_{02} + a_{03} + a_{04} = 2s_0$, $a_{02} + a_{12} + a_{23} + a_{24} = 2s_2$ for some nonnegative integers s_0, s_2 . Then $\prod \phi_{ij}^{a_{ij}} = (t_0^{-2})^{s_0} (t_2^{-2})^{s_2} \prod \bar{\phi}_{ij}^{a_{ij}}$. \square

THEOREM 5.21. *The GIT quotient of $\text{Spec } \mathcal{R}$ associated with the linearization χ is a crepant resolution of singularities of V/G . There are exactly two such resolutions, the other one is the GIT quotient of $\text{Spec } \mathcal{R}$ for the linearization $\chi^{(1,2)}$.*

PROOF. Lemma 5.20 gives a birational morphism $X \rightarrow V/G$. This morphism is projective by [CLS11, Prop. 14.1.12] applied to $\text{Spec } \mathcal{R}$ and its ambient affine space. So it suffices to prove the smoothness of the GIT quotient. By Lemma 5.18, the action of \mathbb{T} on the set of χ -stable points of $\text{Spec } \mathcal{R}$ is free. By Corollary 5.19 we know that the set of χ -stable points of $\text{Spec } \mathcal{R}$ is nonsingular. Now we may deduce

nonsingularity of the quotient since the geometric quotient of a nonsingular variety by a free torus action is nonsingular.

To show that this resolution is crepant it suffices to note that it gives crepant, i.e. minimal, resolution of A_1 -singularities in codimension 2. This is true since generic fibers of the resolution restricted to exceptional divisor are irreducible.

To prove the second part we note that by [WW03, Thm. 1.2] any two such resolutions differ by finite sequence of Mukai flops with respect to projective planes contained in the central fiber. Since we prove in section 5.6 that there is only one such plane, there are exactly two symplectic resolutions. It remains to observe that changing linearization from $\chi^{(2,1)}$ to $\chi^{(1,2)}$ yields a Mukai flop of GIT quotients. \square

5.6. The central fiber. Let $\varphi: X \rightarrow V/G$ be one of the resolutions obtained as (nonsingular) GIT quotients of $\text{Spec } \mathcal{R}$ by Theorem 5.21. We proceed to describe the central fiber $S = \varphi^{-1}([0])$, i.e. the fiber over the image of 0 via the quotient map $V \rightarrow V/G$. It is the unique 2-dimensional fiber of φ . By the McKay correspondence S has two components. Indeed, in $G \cong D_8$ we have two more conjugacy classes apart from two formed by symplectic reflections (corresponding to the components of exceptional divisor), and the class of the identity (corresponding to the whole variety X).

PROPOSITION 5.22. *The central fiber of φ decomposes as $S = F \cup P$ where F is isomorphic to the Hirzebruch surface \mathbb{F}_4 and $P \cong \mathbb{P}^2$. The intersection $F \cap P$ is a smooth quadric on P .*

We will verify this statement directly by describing the preimage of $[0]$ in $\text{Spec } \mathcal{R}$ and then investigating its quotient by \mathbb{T} . Let W be the preimage of $[0]$ under the map $\text{Spec } \mathcal{R} \rightarrow V/G$ induced by the isomorphism $\mathbb{C}[V]^G \cong \mathcal{R}^{\mathbb{T}} \subseteq \mathcal{R}$. If $X \rightarrow V/G$ is one of the resolutions constructed in the previous section, we have the commutative diagram:

$$\begin{array}{ccccc}
 W & \subset & \text{Spec } \mathcal{R} & & \\
 \downarrow & & \downarrow & \nearrow & \\
 [0] & \in & V/G & \xleftarrow{\varphi} & X
 \end{array}$$

(Note: The diagram shows a commutative square with $W \subset \text{Spec } \mathcal{R}$ at the top-left, $[0] \in V/G$ at the bottom-left, X at the bottom-right, and $S \subset X$ at the top-right. A solid arrow points from W to $[0]$. A solid arrow points from $\text{Spec } \mathcal{R}$ to V/G . A dashed arrow points from W to S . A solid arrow points from S to X . A solid arrow labeled φ points from X to V/G .)

LEMMA 5.23. *$W \subseteq \text{Spec } \mathcal{R} \subseteq \mathbb{C}^{12}$ decomposes into irreducible varieties as $W = W_u \cup W_{02} \cup W_0 \cup W_2$, where $\dim W_u = 2$ and $\dim W_{02} = \dim W_0 = \dim W_2 = 4$. More precisely:*

- W_u is a two-dimensional affine subspace of \mathbb{C}^{12} given by $\{w_{ij} = 0: 0 \leq i < j \leq 4\}$.
- W_{02} is a four-dimensional subvariety of \mathbb{C}^{12} with ideal generated by

$$\begin{array}{ll}
 w_{01}w_{23} - w_{03}w_{12} & w_{01}^2 + w_{03}^2 + w_{04}^2 \\
 w_{03}w_{24} - w_{04}w_{23} & w_{12}^2 + w_{23}^2 + w_{24}^2 \\
 w_{04}w_{12} - w_{01}w_{24} & w_{01}w_{12} + w_{03}w_{23} + w_{04}w_{24} \\
 w_{13}, w_{14}, w_{34}, u_0, u_2.
 \end{array}$$

- W_0 is a four-dimensional subvariety of \mathbb{C}^{12} with ideal generated by $w_{01}, w_{03}, w_{04}, w_{13}, w_{14}, w_{34}, u_2, w_{02}^2 u_0 + w_{12}^2 + w_{23}^2 + w_{24}^2$.
- W_2 is a four-dimensional subvariety of \mathbb{C}^{12} with ideal generated by $w_{12}, w_{23}, w_{24}, w_{13}, w_{14}, w_{34}, u_0, w_{02}^2 u_2 + w_{01}^2 + w_{03}^2 + w_{04}^2$.

PROOF. The irreducible components of W are computed in Macaulay2, [GS13], by finding the primary decomposition of the image in \mathcal{R} of the ideal of point $[0] \in V/G$. This is the ideal of \mathcal{R} generated by invariants of the \mathbb{T} -action on \mathcal{R} . \square

In what follows we take the linearization of \mathbb{T} corresponding to $\chi = \chi^{(2,1)}$. Results and proofs in the case of $\chi^{(1,2)}$ are essentially the same. We start from describing the set of χ -stable points of W .

LEMMA 5.24. *The set of χ -semistable points on each component of W coincides with the set of χ -stable points on this component and is equal to its intersection with $(\mathbb{C}^{12})^{ss}$. In particular $W^{ss} = W^s = W_2^s \cup W_{02}^s$.*

PROOF. This follows from the description of χ -stable points in Lemma 5.17. \square

Now we look at the quotient of the set of stable points in W_2 , which gives the component isomorphic to \mathbb{P}^2 .

PROPOSITION 5.25. *The quotient of W_2^s by the action of \mathbb{T} is isomorphic to \mathbb{P}^2 . The quotient of the intersection $W_{02}^s \cap W_2^s$ is mapped by this isomorphism to a smooth quadric in \mathbb{P}^2 .*

PROOF. By Lemma 5.23 we have

$$W_2 \cong \text{Spec } \mathbb{C}[w_{02}, w_{01}, w_{03}, w_{04}, u_2] / (w_{02}^2 u_2 + w_{01}^2 + w_{03}^2 + w_{04}^2).$$

Lemma 5.24 gives an open cover of W_2^s consisting of three open subsets defined by conditions $w_{02}w_{01} \neq 0$, $w_{02}w_{03} \neq 0$, $w_{02}w_{04} \neq 0$. In particular, W_2^s is contained in the affine open subset given by $w_{02} \neq 0$, which is clearly isomorphic to $\text{Spec } \mathbb{C}[w_{02}, w_{01}, w_{03}, w_{04}]_{w_{02}}$.

Consider the ring homomorphism $\mathbb{C}[z_1, z_3, z_4] \rightarrow \text{Spec } \mathbb{C}[w_{02}, w_{01}, w_{03}, w_{04}]_{w_{02}}$ defined by $z_i \mapsto w_{0i}$ for $i = 1, 3, 4$.

It becomes the graded ring homomorphism if we take the standard \mathbb{Z} -grading on $\mathbb{C}[z_1, z_3, z_4]$ associated with the \mathbb{C}^* action by homotheties, the \mathbb{Z}^2 -grading induced by \mathbb{T} -action on the second ring and the monomorphism $\mathbb{Z} \rightarrow \mathbb{Z}^2$ given by $n \mapsto (n, 0)$. Thus the corresponding morphism of varieties is equivariant with respect to considered torus actions.

Moreover, one may verify that it induces the isomorphism of quotients by looking at open covers $\{z_i \neq 0\}$ and $\{w_{0i} \neq 0\}$ for $i = 1, 3, 4$ of sets of stable points. Hence indeed we obtain \mathbb{P}^2 as a quotient.

To prove the second statement, observe that the intersection $W_{02} \cap W_2$ is defined in the open subset $\{w_{02} \neq 0\}$ of W_2 by equation $w_{01}^2 + w_{02}^2 + w_{03}^2 = 0$. This corresponds to the subvariety of \mathbb{P}^2 given by homogeneous equation $z_1^2 + z_3^2 + z_4^2 = 0$ which is a smooth quadric. \square

Finally, we recover the second component, which is present in the central fiber for both constructed resolutions.

PROPOSITION 5.26. *The quotient of W_{02}^s by the action of \mathbb{T} is isomorphic to the Hirzebruch surface \mathbb{F}_4 .*

PROOF. We may consider W_{02} as an affine subvariety of \mathbb{C}^7 with coordinates $w_{02}, w_{01}, w_{03}, w_{04}, w_{12}, w_{23}, w_{24}$. After substituting $w'_{03} = -w_{03} - iw_{04}$, $w'_{04} = w_{03} - iw_{04}$, $w'_{23} = -w_{23} - iw_{24}$, $w'_{24} = w_{23} - iw_{24}$ we see that W_{02} is defined by a binomial ideal generated by

$$\begin{array}{cccc} w_{01}w_{23} - w_{03}w_{12} & w_{03}w_{24} - w_{04}w_{23} & w_{04}w_{12} - w_{01}w_{24} & w_{01}w_{12} - w_{03}w_{24} \\ w_{01}^2 - w_{03}w_{04} & w_{12}^2 - w_{23}w_{24} & & \end{array}$$

where, by abuse of notation, new coordinates are again called w_{ij} . Thus W_{02} is an affine toric subvariety of \mathbb{C}^7 .

It corresponds to the cone $\sigma \subset N_{\mathbb{R}} \simeq \mathbb{R}^4$ (where the one-parameter subgroup lattice N is the standard lattice $\mathbb{Z}^4 \subseteq \mathbb{R}^4$) with rays

$$(0, 1, 1, -1), (0, 1, 2, -1), (0, 0, 0, 1), (0, 1, 0, 1), (1, -1, -1, 1).$$

One may check it by computing the Hilbert basis of the dual cone and the corresponding ideal of relations, e.g. in Macaulay2, [GS13]. Denote the associated big torus by \mathbb{T}_N .

By Lemma 5.24 the complement of the set of χ -stable points on W_{02} is a sum of two irreducible toric, hence also \mathbb{T}_N -invariant, subvarieties

$$Y_1 = V(w_{01}, w_{03}, w_{04}, w_{12}^2 - w_{23}w_{24}), \quad Y_2 = V(w_{02}, w_{12}, w_{23}, w_{24}, w_{01}^2 - w_{03}w_{04})$$

of \mathbb{C}^7 . Note that $\text{codim}_{W_{02}} Y_1 = 1$ and $\text{codim}_{W_{02}} Y_2 = 2$.

Since Y_1, Y_2 are irreducible and \mathbb{T}_N -invariant, they are closures of some orbits O_1, O_2 of \mathbb{T}_N -action on W_{02} . They correspond to faces σ_1, σ_2 of σ of dimension 1 and 2 respectively.

The description of the ideal of the intersection of orbit closure with open affine cover in terms of the fan of a toric variety, see [CLS11, 3.2.7], gives us in particular the conditions

$$\begin{aligned} (1, 1, 0, 0), (1, 0, 1, 0), (1, 2, -1, 0) &\in (\sigma_1^\vee \setminus \sigma_1^\perp), \\ (1, 0, 0, 0), (0, 0, 1, 1), (0, -1, 2, 1), (0, 1, 0, 1) &\in (\sigma_2^\vee \setminus \sigma_2^\perp). \end{aligned}$$

This suffices to deduce that σ_1 is spanned by $(0, 1, 1, -1)$ and σ_2 by $(0, 0, 0, 1)$ and $(1, -1, -1, 1)$.

Now we may describe the fan of the quotient of W_{02}^s by the \mathbb{T} -action as follows (cf. [Ham99, Thm. 5.1]). We take the fan obtained by removing all cones containing either σ_1 or σ_2 from the fan of σ and compute its image under the homomorphism $N_{\mathbb{R}} \rightarrow \mathbb{R}^2$ corresponding to the quotient of \mathbb{T}_N by \mathbb{T} .

On the level of monomial lattices the embedding $\mathbb{T} \cong (\mathbb{C}^*)^2 \subset \mathbb{T}_{W_{02}}$ can be given by characters corresponding to the columns of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 2 \end{pmatrix}.$$

This can be understood by looking at the action of \mathbb{T} in the original coordinates on \mathbb{C}^7 and rewriting it in terms of a basis of the monomial lattice M of \mathbb{T}_N .

Thus on the level of one-parameter subgroup lattices we take the homomorphism given by a basis of the kernel of the matrix above:

$$\begin{pmatrix} 0 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

One can compute now that the images of faces of σ not containing σ_1 nor σ_2 form a complete fan in \mathbb{Z}^2 with rays $(1, 1), (1, 0), (-1, 3), (-1, 0)$, which is the fan of \mathbb{F}_4 . \square

Proposition 5.22 is thus a direct consequence of Propositions 5.25 and 5.26.

6. Binary tetrahedral group G_4

Consider the symplectic representation $G = G_4 \subset \mathrm{Sp}(\mathbb{C}^4, \omega)$ of the binary tetrahedral group which is generated by the following two matrices:

$$\iota = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \tau = -\frac{1}{2} \begin{pmatrix} (1+i)\epsilon & (-1+i)\epsilon & 0 & 0 \\ (1+i)\epsilon & (1-i)\epsilon & 0 & 0 \\ 0 & 0 & (1+i)\epsilon^2 & (-1+i)\epsilon^2 \\ 0 & 0 & (1+i)\epsilon^2 & (1-i)\epsilon^2 \end{pmatrix},$$

where ϵ is a 3rd root of unity and $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ (see [LS12]). We sum up the basic properties of this representation in the next proposition.

PROPOSITION 6.1.

- The commutator subgroup $H = [G, G]$ is isomorphic to quaternion group Q_8 and is generated by matrices ι, ξ , where

$$\xi = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

In particular it does not contain any symplectic reflection.

- The quotient group $\mathrm{Ab}(G) = G/H$ is isomorphic to the group \mathbb{Z}_3 and is generated by the image of τ .
- There are two four-elements conjugacy classes of symplectic reflections, represented by $T_1 = \tau$ and $T_2 = \tau^2$. Hence by the McKay correspondence there are two components E_1, E_2 of the exceptional divisor corresponding to conjugacy classes of symplectic reflections T_1, T_2 respectively.
- The singular locus of the quotient V/G is irreducible and consists of one transversal A_2 singularity since the symplectic reflection τ is of order 3 and there are no reflections in the commutator subgroup (see Remark 3.2). In particular the intersection matrix $(E_i \cdot C_j)_{i,j}$ is a Cartan matrix of type A_2 :

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$$

- Let ν_i be the monomial valuation $\mathbb{C}(V)^{[G,G]} \rightarrow \mathbb{Z}$ associated with symplectic reflection T_i . Then $\nu_i|_{\mathbb{C}(V)^G} = 3\nu_{E_i}$, $i = 1, 2$, by Lemma 2.12.

eigenvalue 1:

$$\begin{aligned}
\phi_1 &= y_1 x_2 - x_1 y_2 \\
\phi_2 &= x_2^5 y_2 - x_2 y_2^5 \\
\phi_3 &= x_1^5 y_1 - x_1 y_1^5 \\
\phi_4 &= x_1^4 + (-4a^2 + 2)x_1^2 y_1^2 + y_1^4 \\
\phi_5 &= x_2^4 + (4a^2 - 2)x_2^2 y_2^2 + y_2^4 \\
\phi_6 &= x_1 x_2^3 + (-2a^2 + 1)y_1 x_2^2 y_2 + (-2a^2 + 1)x_1 x_2 y_2^2 + y_1 y_2^3 \\
\phi_7 &= x_1^3 x_2 + (2a^2 - 1)x_1 y_1^2 x_2 + (2a^2 - 1)x_1^2 y_1 y_2 + y_1^3 y_2 \\
\phi_8 &= (2a^2 - 2)x_1^2 y_1 x_2^3 + (2a^2 - 2)x_1^3 x_2^2 y_2 + (-2a^2 + 2)y_1^3 x_2 y_2^2 + (-2a^2 + 2)x_1 y_1^2 y_2^3
\end{aligned}$$

eigenvalue ϵ :

$$\begin{aligned}
\phi_9 &= -a^2 x_1^2 x_2^2 + (-(1/3)a^2 + 2/3)y_1^2 x_2^2 + (-(4/3)a^2 + 8/3)x_1 y_1 x_2 y_2 + \\
&\quad + (-(1/3)a^2 + 2/3)x_1^2 y_2^2 - a^2 y_1^2 y_2^2 \\
\phi_{10} &= x_2^4 + (-4a^2 + 2)x_2^2 y_2^2 + y_2^4 \\
\phi_{11} &= x_1^3 x_2 + (-2a^2 + 1)x_1 y_1^2 x_2 + (-2a^2 + 1)x_1^2 y_1 y_2 + y_1^3 y_2 \\
\phi_{12} &= -10x_1^4 y_1 x_2 + 2y_1^5 x_2 - 2x_1^5 y_2 + 10x_1 y_1^4 y_2 \\
\phi_{13} &= 2x_1 y_1 x_2^4 + 4x_1^2 x_2^3 y_2 - 4y_1^2 x_2 y_2^3 - 2x_1 y_1 y_2^4
\end{aligned}$$

eigenvalue ϵ^2 :

$$\begin{aligned}
\phi_{14} &= (a^2 - 1)x_1^2 x_2^2 + ((1/3)a^2 + 1/3)y_1^2 x_2^2 + ((4/3)a^2 + 4/3)x_1 y_1 x_2 y_2 + \\
&\quad + ((1/3)a^2 + 1/3)x_1^2 y_2^2 + (a^2 - 1)y_1^2 y_2^2 \\
\phi_{15} &= x_1^4 + (4a^2 - 2)x_1^2 y_1^2 + y_1^4 \\
\phi_{16} &= x_1 x_2^3 + (2a^2 - 1)y_1 x_2^2 y_2 + (2a^2 - 1)x_1 x_2 y_2^2 + y_1 y_2^3 \\
\phi_{17} &= 2y_1 x_2^5 + 10x_1 x_2^4 y_2 - 10y_1 x_2 y_2^4 - 2x_1 y_2^5 \\
\phi_{18} &= -4x_1^3 y_1 x_2^2 - 2x_1^4 x_2 y_2 + 2y_1^4 x_2 y_2 + 4x_1 y_1^3 y_2^2
\end{aligned}$$

TABLE 3. Generators of $\mathbb{C}[V]^H$ that are eigenvectors of $\text{Ab}(G)$ (a is a 12th root of unity).

This allows us to compute a set of generators of $\mathbb{C}[V]^{[G,G]}$ consisting of eigenvectors of the $\text{Ab}(G)$ action. The method of the proof is the same as in the case of S_3 , cf. Proposition 4.2.

PROPOSITION 6.2. *The elements ϕ_i in the Table 3 are generators of the ring of invariants $\mathcal{P} = \mathbb{C}[V]^H$, which are eigenvectors of the action of τ .*

As in the above cases, we expect that the generating set of the Cox ring of a symplectic resolution can be described as in Theorem 3.8.

CONJECTURE 6.3. *The image of the Cox ring $\Theta(\mathcal{R}(X)) \subseteq \mathcal{P}[t_1^{\pm 1}, t_2^{\pm 1}]$ is generated by*

$$t_1^{-2} t_2, t_1 t_2^{-2}, \phi_1, \dots, \phi_8, \phi_9 t_2, \dots, \phi_{13} t_2, \phi_{14} t_1, \dots, \phi_{18} t_1.$$

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